Department of Mathematics and Statistics

Junior Mathematics Competition 2021

Student Report

Year 9 (Form 3) Prize Winners

First	Gary Yuan	Auckland Grammar
Second	Yu-Jui Chu	Auckland Grammar
Third	Daniel Xian	St Kentigern College

Top 30 (in School Order):

Allen Weng, ACG Parnell College
Benjamin Kuo, ACG Strathallan College
Boning Dai, Auckland Grammar
Jessica Rankin, Burnside High School
Kenyan Zong, Hutt International Boys' School
Ray Shi, Kings College
Alyxia Jaccard, Kristin School
Justin Cui, Macleans College
Howard Shang, Macleans College
Josephine Sim, Macleans College
William Wang, Macleans College
Nancy Zhang, Macleans College
Pratyush Khatiwada, Mount Roskill Grammar School
Sia Goel, Newlands College
Edward Xiao, Rangitoto College
Yasha Potanin, Scots College
Tony Yu, Scots College
Ellie Siu, St Cuthbert's College
Ruby Goodchild, St Hilda's Collegiate
Oscar Prestidge, St Kentigern College
Preston Tee, St Kentigern College
Leon Lee, St Peter's School (Cambridge)
Leon Dromer, Takapuna Grammar School
David Xie, Takapuna Grammar School
Zeph Zhong, Takapuna Grammar School
Angela Lin, Wentworth College
Elaine Zhou, Westlake Girls' High School

Year 10 (Form 4) Prize Winners

First	Andrew Chen	Macleans College
Second	Nico McKinlay	St Kentigern College
Third	Dongwook Oh	Auckland Grammar

Top 30 (in School Order):

Nicholas Chen, ACG Parnell College
Pratham Shah, ACG Parnell College
Amanjot Kaur, ACG Strathallan College
Chris Pan, Auckland Grammar
Seivin Kim, Avondale College
David Evans, Canterbury Home Educators
Toby Churchman, Garin College
Adrien Auvray Matyn, Kaikorai Valley College
Connor Gray, King's High School
Evan Huang, Lynfield College
Yixue Wang, Macleans College
Emily Yin, Macleans College
Jesse Zhang, Macleans College
Raymond Zhang, Macleans College
Jessica Huang, Mount Roskill Grammar School
Christopher Mara, Mount Roskill Grammar School
Megan Sushames, Otumoetai College
Allen Li, Rangitoto College
Albin Paulson, Sancta Maria College
Bryan Cooper, St Andrew's College
Vanessa Bu, St Cuthbert's College
Ena Yin, St Cuthbert's College
Jonathan Chia, St Kentigern College
Rune Nicholson, Wellington High School
Richard Meng, Westlake Boys' High School
Minjae Park, Westlake Boys' High School
Ubeen Sim, Westlake Boys' High School

Year 11 (Form 5) Prize Winners

First	Bufan Zhao	ACG Parnell College
Second	Eric Liang	ACG Parnell College
Third	Oliver Gunson	Auckland Grammar

Top 30 (in School Order):

Harit Patel, ACG Strathallan College
Jayden Kris Dylan Kumar, Auckland Grammar
Daniel Park, Auckland International College
Yifei Song, Auckland International College
Annabelle Brownsword, Burnside High School
Johnny Lawrey, Burnside High School
Devin Lin, Burnside High School
Jimmy Dawson, Christchurch Boys' High School
Coeun Ham, Epsom Girls' Grammar School
Emma Ying, Kings College
Sam Zhang, Kristin School
Alexander Sun, Logan Park High School
Valerie Lau, Macleans College
Eric Lee, Macleans College
Melinda Shao, Macleans College
Jonathan Siah, Rangitoto College
Jason Yang, Scots College
Gemma Lewis, St Andrew's College
Kotori Mori, St Andrew's College
Lingshi Chen, St Cuthbert's College
Grace Wu, St Cuthbert's College
Sean Wang, St Kentigern College
Jolin Yang, St Paul's Collegiate School
Joe McKibbin, Takapuna Grammar School
Daniel Nelson, Tauranga Boys' College
Sarthak Singh, Wellington College
Alex Vautier, Wellington College

2021 Model Solutions

As always only one method is shown. Often several methods exist. We don't guarantee that the method shown is the best or fastest.

Question 1: 10 marks (Years 9 and below only)

You do not need to show working in this question. Note that a perfect square is of the form $n \times n$ and a cube is of the form $n \times n \times n$, where n is an integer in both cases.

If $a = 1$, $b = 2$, $c = 3$, and so on up to $z = 26$, use the following clues to determine a mystery word. For each clue write down the associated number. At the end write down your word. (All letters map to a number between 1 and 26 inclusive.)

(a) My first letter is equal to the positive square root of 49.

■ $7 \times 7 = 49$, hence 7 or 'g'.

(b) My second and fifth letters are both equal to the smallest sum of two different primes.

■ $2 + 3 = 5$, hence 5 or 'e'.

(c) My third letter is equal to three times my second letter.

■ $3 \times 5 = 15$, hence 15 or 'o'.

(d) My fourth letter is equal to the sixth prime number.

■ The sixth prime number is 13, or 'm'.

(We then repeat 'e' in our word.)

(e) My sixth letter is greater than my seventh number, smaller than 26, and is equal to a prime times a perfect square.

■ $5 \times 2^2 = 20 > 18$, and $7 \times 2^2 = 28 > 26$, hence 20 or 't'.

(f) My seventh letter is equal to twice the square of my ninth letter.

■ $2 \times 3^2 = 18$, hence 18 or 'r'.

(g) My eighth letter is equal to the first positive odd number one greater than a non-zero positive cube number.

■ $1^3 + 1 = 2$ is even, so $2^3 + 1 = 9$ is what we want. So 9 or 'i'.

(h) My ninth letter is equal to the first odd prime number.

■ The first odd prime number is 3, or 'c'.

■ The word is 'geometric'.

This was generally well done. Most students were able to get most (if not all) of the clues correct.

Question 2: 10 marks (Years 10 and below only)

Jane is building her dream home. She has a section of 500 square metres on which to build. She considers two plans for her house:

> *(1) A three storey house with a ground floor area of 380 square metres.*
>
> *(2) A two storey house with a ground floor area that covers 87.5% of her section.*

(a) What percentage of Jane's section will plan (1) take up?

■ $(380/500) \times 100 = 76$ percent.

(b) What will the ground floor area of Jane's house be in square metres if she chooses plan (2)? Your answer should be given to two decimal places.

■ $(87.5/100) \times 500 = 437.50$ square metres.

(c) Suppose Jane chooses plan (1) for her house. If the first floor is 75% of the area of the ground floor and the second floor is 75% of the area of the first floor, what is the area of the second floor in square metres? Your answer should be given to two decimal places.

■ $0.75 \times 0.75 \times 380 = 213.75$ square metres.

(d) If the first floor of the house in plan (2) has the same floor area as the ground floor, will the total floor area (the sum of the floor area of each floor) be bigger in plan (1) or plan (2)? Write the total floor area for both plans as part of your answer.

■ Plan (1) has a total floor area of $0.75 \times 0.75 \times 380 + 0.75 \times 380 + 380 = 878.75$ square metres, while plan (2) has a total floor area of $2 \times 437.50 = 875.00$ square metres, so plan (1) has a bigger floor area.

Like Question 1 the majority of students had little problem with this question. The hardest part was part (d), where some students could not calculate the area of plan (1).

Question 3: 20 marks (All Years)

At Kakanui University a regular undergraduate degree takes three academic years. Each paper in the degree is worth a different number of points (either 5 points, 9 points, or 12 points), with the final degree being worth a total of 180 points.

(a) If each academic year is made up of two *semesters*, what is the average number of points needed per semester to finish the undergraduate degree in three years?

■ $180/(2 \times 3) = 30$ points per semester.

(b) If each academic year is instead made up of three *trimesters*, what is the average number of points needed per trimester to finish the undergraduate degree in three years?

■ $180/(3 \times 3) = 20$ points per trimester.

For the rest of the question we assume each academic year is made up of two semesters. Each semester students must do a minimum of 24 points worth of papers, and a maximum of 32 points worth of papers, and each student must do papers every semester in the three academic years.

(c) If a student does exactly one semester of 24 points in a three year undergraduate degree, what is the average number of points that student must do in the rest of the semesters needed to finish the degree?

■ $(180 - 24)/5 = 31.2$ points per semester.

(d) What is the maximum number of semesters with precisely 32 points in them that a student may have in a three year undergraduate degree?

■ $(180 - 5 \times 32) = 20$ points for the last semester, which is too few, but $(180 - 4 \times 32) = 52$ which allows (for example) a semester of 24 points and a semester of 28 points to make up the difference. Hence 4 semesters of 32 is the maximum.

(e) What is the maximum number of 9 point papers a student may take in a three year undergraduate degree?

■ Seventeen.

We have six semesters in total. Each semester can have at most three 9 point papers, which means a three year degree can have at most 3×6 eighteen 9 point papers. (You can't have x 9 point papers where $x \geq 19$.)

If we have three 9 point papers every semester, then the 9 point papers over the three years total $9 \times 3 \times 6 = 162$ points, 18 points short of 180. Since we can't have any more 9 point papers in the degree and there is no combination of 5 and 12 point papers that sum to 18 points, there must be at least one semester with two or fewer 9 point papers. This means we cannot have eighteen 9 point papers.

If we have five semesters with three 9 point papers, and one semester with two 9 point papers (so seventeen 9 point papers in total), then we are short 27 points. $27 = 12 + 5 + 5 + 5$. We can do one 12 point paper in the semester with two 9 point papers (making 30 points in that semester), and three of the other semesters can have one 5 point paper added (making 32 points in each case). Thus seventeen 9 point papers works.

This ended up being the hardest question that students from all levels could answer, which meant that its placement as Question 3 was a little unfortunate. Students found the first three parts easy enough, but struggled with the intricacies of the latter two sections. The key to getting high marks in parts (d) and (e) was to realise that one had to show not only that a certain number of semesters or papers worked, but also that* no higher number of semesters or papers could work. *Showing both was beyond all but a few, especially in part (e).

Question 4: 20 marks (All Years)

In this question the *resolution* of a rectangular screen or photo is written as $w \times h$, where w is the width in pixels and h is the height in pixels. For example a screen of resolution 800×600 is 800 pixels wide and 600 pixels high, and has a total of 480000 pixels.

Achara is designing a photo editing website. Some photos have to be scaled to a certain width and height to fit on the various screen sizes Achara anticipates people will use. (Photos that are smaller than the resolution of a given screen do not need to be scaled.) The aspect ratio of each photo will be preserved in all cases.

For a photo of resolution $a \times b$ and (smaller) screen of resolution $c \times d$, the *scale factor* s is the largest number between 0 and 1 such that $as \leq c$ and $bs \leq d$. For example if a photo of resolution 3840×2160 is to be scaled to fit a screen of resolution 1920×1080, the *scale factor* s will be 0.5.

(a) How many pixels are there in a screen of resolution 1280×1024?

■ $1280 \times 1024 = 1310720$ pixels.

(b) Consider a photo of resolution 1920×1080.

(i) What scale factor s is needed to scale our photo to fit a screen of resolution 1280×720? Round your answer to three decimal places.

■ $1280/1920$ and $720/1080$ both equal .667 (when rounded to three decimal places).

(ii) What scale factor s is needed to scale our photo to fit a screen of resolution 1024×768? Round your answer to three decimal places.

■ To three decimal places $1024/1920 = .533$ and $768/1080 = .711$. Since $.711 \times 1920 > 1024$, $s \neq .711$, so $s = .533$.

Achara decides that putting a small border of 10 pixels around each photo will look better on a given screen.

(c) If a photo of resolution 1920×1080 is scaled to fit a screen of resolution 1024×768 and a border of 10 pixels is required, to three decimal places what is the scale factor needed?

■ A border of 10 pixels means that we need to subtract 20 from our width and 20 from our height. So we need to scale our photo to a resolution of 1004×748.

Then s is either $1004/1920 = 0.523$ or $748/1080 = 0.693$. Like in (b)(ii) we choose the smaller value, so $s = 0.523$.

One of the ways people can edit photos is to crop them — that is, they make a smaller photo by selecting a particular rectangular part of the larger photo.

(d) If a photo of resolution 1920×1080 is to be cropped to a resolution of 1024×768, how many pixels will be removed in total?

■ The original photo contains $1920 \times 1080 = 2073600$ pixels, while the cropped photo contains $1024 \times 768 = 786432$ pixels.

Thus the number of pixels removed is $2073600 - 786432 = 1287168$ pixels.

Students found this question easier than Question 3 on the whole, although the average mark was lower here than in the following Question 5. The first and last parts of the question were reasonably well done; it was the middle parts of the question where students struggled. In part (b)(ii) quite a few students failed to realise the ratios of the two dimensions were different (they were deliberately the same in part (b)(i)). The most common error in part (c) was removing only 10 pixels from the width and 10 pixels from the height of the screen; other students added pixels to the width and height instead, or changed the initial photo's resolution, both of which produced subtly incorrect results.

Question 5: 20 marks (All Years)

Let $\sigma(n)$ be the sum of the divisors of a natural number n (the divisors include 1 and n). For example, if $n = 14$ then $\sigma(14) = 1 + 2 + 7 + 14 = 24$.

(a) For the following n find $\sigma(n)$:

(i) $n = 6$.

■ 6 has 1, 2, 3, and 6 as factors, so $\sigma(6) = 1 + 2 + 3 + 6 = 12$.

(ii) $n = 13$.

■ 13 has 1 and 13 as factors, so $\sigma(13) = 1 + 13 = 14$.

(iii) $n = 30$.

■ 30 has 1, 2, 3, 5, 6, 10, 15, and 30 as factors, so $\sigma(30) = 1 + 2 + 3 + 5 + 6 + 10 + 15 + 30 = 72$.

We call a natural number d *deficient* when $\sigma(d) < 2d$, while a *perfect* natural number r has $\sigma(r) = 2r$, and an *abundant* natural number a has $\sigma(a) > 2a$.

(b) Find the smallest deficient number greater than 30.

■ 31 has 1 and 31 as factors, so $\sigma(31) = 1 + 31 = 32 < 2 \times 31$, so 31 is the smallest deficient number greater than 30.

(c) Find the largest perfect number smaller than 30.

■ 29 has 1 and 29 as factors, so $\sigma(29) = 1 + 29 = 30 \neq 2 \times 29$, so 29 is not a perfect number. 28 has 1, 2, 4, 7, 14, and 28 as factors, so $\sigma(28) = 1 + 2 + 4 + 7 + 14 + 28 = 56 = 2 \times 28$, so 28 is a perfect number. Since 29 is not a perfect number 28 is the the largest perfect number smaller than 30.

(d) Find an abundant number between 41 and 50 inclusive.

■ One (or both) of 42 or 48:

42 has 1, 2, 3, 6, 7, 14, 21, and 42 as factors, so $\sigma(42) = 1 + 2 + 3 + 6 + 7 + 14 + 21 + 42 = 96 > 2 \times 42$, so 42 is an abundant number.

48 has 1, 2, 3, 4, 6, 8, 12, 16, 24, and 48 as factors, so $\sigma(48) = 1+2+3+4+6+8+12+16+24+48 = 124 > 2 \times 48$, so 48 is an abundant number.

All natural numbers can be classified as either deficient, perfect, or abundant.

(e) Briefly explain why every prime number is deficient.

■ Every prime number has exactly two factors, itself and 1. So for a prime number p we have $\sigma(p) = 1 + p < 2p$ (since $1 + n >= 2n$ means $n <= 1$ and 1 is not a prime number).

(f) For each number between 21 and 30 inclusive, specify if it is deficient, perfect, or abundant. (Include 21 and 30 in your list; you do not need to show working in this part.)

■ Each number is classified as follows:

21 is deficient ($\sigma(21) = 1 + 3 + 7 + 21 = 32 < 2 \times 21$).

22 is deficient ($\sigma(22) = 1 + 2 + 11 + 22 = 36 < 2 \times 22$).

23 is deficient (it is a prime number).

24 is abundant ($\sigma(24) = 1 + 2 + 3 + 4 + 6 + 8 + 12 + 24 = 60 > 2 \times 24$).

25 is deficient ($\sigma(25) = 1 + 5 + 25 = 31 < 2 \times 25$).

26 is deficient ($\sigma(26) = 1 + 2 + 13 + 26 = 42 < 2 \times 26$).

27 is deficient ($\sigma(27) = 1 + 3 + 9 + 27 = 40 < 2 \times 27$).

28 is perfect (from part (c)).

29 is deficient (it is a prime number).

30 is abundant (from part (a)(iii)).

This question was far better answered than the previous two questions. The abstract nature of the question did not seem to scare many students off. It was common to see students get full or near full marks here (the average mark in the question was above 10); the most frequent error amongst top candidates was missing a statement in part (c) to the effect that 29 was not a perfect number. (In some ways this was a similar technicality to those seen in Question 3.)

Question 6: 20 marks (All Years)

For the purposes of this question, assume that a leap year occurs every four years, and such years have 366 days in them, with a day added in February. (Years that are not leap years have 365 days in them.) 2020 is a leap year. Also note that $7 \times 52 = 364$.

Rawiri (whose birthday is in June) goes to the gym every Tuesday and Thursday. Assume that he lives for a very long time.

(a) In 2019 Rawiri went to the gym on a Tuesday on his birthday in June. In what year could he first go again to the gym on a Tuesday which is his birthday?

■ In 2020 Rawiri's birthday is on a Thursday. In 2021 his birthday is on a Friday. In 2022 his birthday is on a Saturday, In 2023 his birthday is on a Sunday. In 2024 (a leap year) his birthday is on a Tuesday again.

(b) In 2020 Rawiri went to the gym on a Thursday on his birthday in June. In what year could he first go again to the gym on a Thursday which is his birthday?

■ From the above Rawiri's birthday is on a Tuesday in 2024, so in 2026 it would be on a Thursday.

(c) In what two consecutive years (after 2020) could Rawiri first go again to the gym on a Tuesday birthday and then on a Thursday birthday the year after?

■ We need another Thursday birthday in a leap year. From the above it can be seen that every 4 years Rawiri's birthday moves forward 5 days. As such Rawiri's birthday in 2028 is on a Sunday, in 2032 it's on a Friday, in 2036 it's on a Wednesday, in 2040 it's on a Monday, in 2044 it's on a Saturday, and in 2048 it's on a Thursday. So the consecutive years we need are 2047 and 2048.

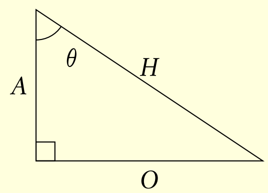
Quite unintentionally this question became a classic example of brute-force being the easiest way forward. It was generally easier (and clearer to the markers) to explain your answers here by just listing years and days of the week, rather than coming up with a complicated mathematical approach without adequate explanation as to what was going on. As such students with a less rigorous mathematical background actually did better than expected, while some more formally trained students tried a more 'elegant' approach which was often not to their benefit.

Question 7: 10 marks (Years 10 and 11 only)

Suppose for some θ where $0° < \theta < 90°$ there exists a real number m such that $\tan\theta = \dfrac{(1+m)}{(1-m)}$.

(a) Find $\cos\theta$ in terms of m.

- We construct the following triangle:



In the above diagram we have $A = (1-m)$ and $O = (1+m)$. Then $\cos\theta = \dfrac{A}{H}$ and

$$
\begin{aligned}
H^2 &= (1-m)^2 + (1+m)^2 \\
&= (1 - 2m + m^2 + 1 + 2m + m^2) \\
&= 2m^2 + 2 && \text{(expand and simplify)} \\
H &= \sqrt{2(1+m^2)} && \text{(or equivalent)}
\end{aligned}
$$

So $\cos\theta = \dfrac{1-m}{\sqrt{2(1+m^2)}}$ or equivalent.

(b) Find the allowable values of m.

- If $0° < \theta < 90°$ then $\tan\theta > 0$. Thus $\dfrac{(1+m)}{(1-m)} > 0$ as well, so $1+m$ and $1-m$ are non-zero real numbers that are either both positive or negative.

This means that either $1+m < 0$ and $1-m < 0$, or $1+m > 0$ and $1-m > 0$.

In the former case we have $m < -1$ and $m > 1$, a contradiction.

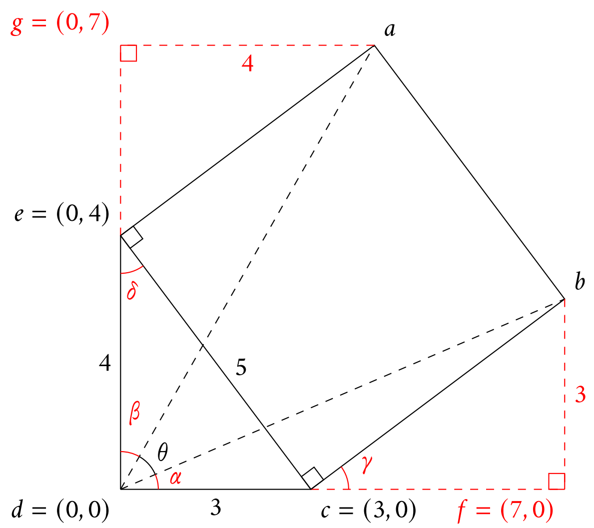
In the latter case we have $m > -1$ and $m < 1$, which is possible. Combining the two statements we get $-1 < m < 1$.

This question lies in stark contrast to Question 6. Only those with a solid grounding in mathematics had any hope of prospering here. In part (a) the approach listed above by far the most common approach, although other methods of finding $\cos\theta$ ***were seen. Part (b) was much less well answered — although the solution above was the most common correct one seen, using the value of*** $\cos\theta$ ***found in (a) is probably a better way to go (as the denominator of the right hand side of the value of*** $\cos\theta$ ***is always greater than 0).***

Question 8: 10 marks (Year 11 only)

It was a nice day, so Scott decided to take his class of gifted students to the gardens for a geometry lesson. He sat his class down at point d, from which everyone could see four fine trees located at a, b, c, and e, which defined the corners of a grass square with side length 5 metres.

■ Note: the diagram has been annotated in red with additional information not seen in the competition paper. Since $\overline{cb}$ has a length of 5 metres, the triangle $\triangle bfc$ is right-angled, and the angle γ has the same magnitude as δ, it follows that the length of $\overline{bf}$ (and $\overline{eg}$) is 3 metres, and the length of $\overline{cf}$ (and $\overline{ag}$) is 4 metres.



(a) If Scott asks his class to find as co-ordinates the points a and b, what do they write down?

■ $a = (4, 7)$ and $b = (7, 3)$.

(b) From the group's vantage point of d, what is the angle θ between a and b? Round your answer to three decimal places.

■ $\tan\alpha = \frac{3}{3+4}$ and $\tan\beta = \frac{4}{3+4}$, so $\theta = 90 - \tan^{-1}\frac{3}{7} - \tan^{-1}\frac{4}{7} = 37.057$ degrees.

(c) Find the area of the triangle defined by a, b, and d

■ The length of $\overline{db}$ is $\sqrt{3^2+7^2}$, while the length of $\overline{da}$ is $\sqrt{4^2+7^2}$. The area of $\triangle abd$ is $\frac{1}{2}\overline{da}\,\overline{db}\sin\theta$, or $\frac{1}{2}(\sqrt{4^2+7^2})(\sqrt{3^2+7^2})(0.603) = 18.5\ \text{m}^2$.

This question was not well done overall, although it is never the intent of the competition setters to make the last question easy. It was perhaps a little disappointing to see so few students get part (a) correct - it was designed to be the 'easy' part of the question. In part (b) a lot of students opted to use the cosine rule to find θ, which involved many more lines of answer than the simpler approach seen here. It was pleasing to see that most students who got part (c) correct quickly realised that the answer is exactly 18.5 m^2.