Department of Mathematics and Statistics

UNIVERSITY of OTAGO
Te Whare Wānanga o Otāgo
NEW ZEALAND

Junior Mathematics Competition

The University of Otago Junior Mathematics Competition 2020

Year 9 (Form 3) Prize Winners

First	Ena Yin	St Cuthbert's College
Second Equal	Belle Yin	St Cuthbert's College
Third Equal	Allen Li	Rangitoto College

Top 30 (in School Order):

Justin Huang, Botany Downs Secondary College
Daniel Lough, Burnside High School
David Evans, Canterbury Home Educators
Thulani Wanninayake Mudiyanselage, Epsom Girls' Grammar School
Winston Weng, Kings College
Nicole Wong, Kristin School
Kenny Jeong, Middleton Grange School
Dawn Chen, St Cuthbert's College
Jennifer Liu, St Cuthbert's College
Jifei Shao, St Cuthbert's College
William Guan, St Kentigern College
Emma Ying, St Kentigern College
Devin Shen, Takapuna Grammar School
Raghav Bhutani, Burnside High School
Haasini Mantena, Burnside High School
Emma Jia, Carmel College
Xinyi Zhang, Glen Eden Intermediate School
Ryan Fan, Kristin School
Stuti Tiwari, Lincoln High School
Ella Cham, St Cuthbert's College
Priyanka Gai, St Cuthbert's College
Sophie Robb, St Cuthbert's College
Yicky Zhou, St Cuthbert's College
Jayden Tee, St Kentigern College
Bill An, Takapuna Grammar School
Julianna Wang, Waikato Diocesan School for Girls
Makaela Cheung, Wellington Girls' College

Year 10 (Form 4) Prize Winners

First	Grace Wu	St Cuthbert's College
Second	Eliza Chi	Epsom Girls' Grammar School
Third	Leo Lai	Elim Christian College

Top 30 (in School Order):

Nikhil Karthik, Botany Downs Secondary College
Woojin Song, Burnside High School
Angela Bi, Epsom Girls' Grammar School
Coeun Ham, Epsom Girls' Grammar School
Laura Li, Epsom Girls' Grammar School
Brian Feng, Kings College
Derek Peng, Kings College
Aaron Shi, Kristin School
Sam Zhang, Kristin School
Abby Campbell, Rangitoto College
Shufei Wu, Rangitoto College
Sophia Zhou, St Cuthbert's College
Eike Flath, Tokomairiro High School
Annabelle Brownsword, Burnside High School
Elaina Fu, Columba College
Charlotte Chan, Epsom Girls' Grammar School
Cindy Kim, Epsom Girls' Grammar School
Theibana Vignakumar, Epsom Girls' Grammar School
Ethan Huang, Kings College
Dianne Lee, Kristin School
Tongsheng Wu, Kristin School
Alexander Sun, Logan Park High School
Jonathan Siah, Rangitoto College
Leonore Li, St Cuthbert's College
Albert Wang, St Peter's College (Epsom)
Debbie Lee, Westlake Girls' High School
Stephanie Zhou, Westlake Girls' High School

Year 11 (Form 5) Prize Winners

First	Hellen Ding	Epsom Girls' Grammar School
Second	Sun-woong Kang	Rangitoto College
Third	Logan McDonald	Tauranga Boys' College

Top 30 (in School Order):

Ian Han, Burnside High School
Kyubeen Kim, Epsom Girls' Grammar School
Janet Guo, Hillcrest High School
Ruilai Ma, John McGlashan College
Wesley Ngo, Kings College
Selwyn Liu, Kristin School
Vito Zou, Kristin School
Benjy Smith, Onslow College
Sawooly Li, Rangitoto College
Benjamin Guo, Riccarton High School
Brena Merz, St Cuthbert's College
Yixiong Hao, St Kentigern College
Yushin Doh, Wellington Girls' College
Ritvik Sharma, Burnside High School
Nancy Wei, Epsom Girls' Grammar School
Martin Brook, John McGlashan College
Bailey Liu, Kings College
Belle Li, Kristin School
He (James) Xu, Kristin School
Caleb Giddy, Middleton Grange School
Zhinuo Huang, Rangitoto College
Rihoko Suzuki, Rangitoto College
Maria Gong, St Cuthbert's College
Charlotte Wen, St Cuthbert's College
Ashley Wang, St Kentigern College
Ayane Kondo, Westlake Girls' High School
Raeanne Leow, Westlake Girls' High School

2020 Model Solutions

As always only one method is shown. Often several methods exist. We don't guarantee that the method shown is the best or fastest.

Question 1: 10 marks (Years 9 and below only)

Mr Quick, a dangerous driver, was driving 660 km from Auckland to Wellington in just 6 hours. This required 62.7 litres of petrol, and when he arrived his tank was empty.

(a) What was his average speed? **The average speed is given as distance over time: speed = $\frac{660\,\text{km}}{6\,\text{h}} = 110\,{}^{\text{km}}\!/_{\text{h}}$.**

(b) How much petrol on average was needed per 100 km? **He used 62.7 litres for 660 km, and hence $\frac{62.7}{660} \times 100 = 9.5\,{}^{\text{litres}}\!/_{100\,\text{km}}$.**

(c) With an average speed of 90 km/h, the car would only have used 8.5 litres of petrol per 100 km. What additional distance could Mr Quick drive with the saved petrol at this speed? (Round your answer to the nearest integer.)

Instead of 62.7 litres, he would have used

$$\begin{aligned} 660\,\text{km} \times \frac{8.5\,\text{litres}}{100\,\text{km}} &= 6.6 \times 8.5\,\text{litres} \\ &= 56.1\,\text{litres} \end{aligned}$$

i.e. 6.6 litres less. With this petrol, he could drive another

$$\frac{6.6\,\text{litres}}{8.5\,{}^{\text{litres}}\!/_{100\text{km}}} \approx 77.65\,\text{km}$$

i.e. about 78 km.

Parts (a) and (b) were generally well done. A lot of students made progress on part (c), but stumbled after reaching 56.1 litres. Other candidates forgot to round to 78 kilometres, which in an easy competition cost them crucial marks.

Question 2: 10 marks (Years 10 and below only)

Methane is a greenhouse gas that is about 50 times more destructive to the environment than carbon dioxide. In New Zealand, methane from cattle digestion accounts for almost one third of the country's greenhouse gas emissions. An average cow releases about 100 kg of methane each year. Recently, a New Zealand science organisation started experimenting with a food supplement from a native seaweed. First tests show that just a small quantity of this supplement could reduce stock methane emissions by 80%!

(a) New Zealand cows release about 500 million kg methane each year. How many cows do we approximately have in New Zealand?

We need to divide the methane released each year (500 million kg) by the amount released by a single cow (100 kg), so $\frac{500\,\text{million kg}}{100\,\text{kg}} = 5$ million cows.

(b) If every cow in New Zealand received the food supplement, how many kilograms of methane less would be burped out each year?

The supplement would reduce the methane release by 80%. Hence the total methane reduction per year would be

$$\begin{aligned} 80\% \times 500\,\text{million kg} &= \frac{80}{100} \times 500\,\text{million kg} \\ &= 400\,\text{million kg}. \end{aligned}$$

(c) In order to supply the growing world population with dairy and beef, even more cows will be needed in the future. However, with the food supplement New Zealand could have more cows and still reduce the release of methane into the environment. How many cows could we have if all of them received the food supplement and we wanted to reduce the methane emissions to 30% of the current value?

With the current number of cows, the supplement would reduce the annual methane release to 20% of the current value. If we are happy with 30%, then we can have $30/20 = 3/2 = 1.5$ times the current number of cows.

Since there are now about 5 million cows, we could have $1.5 \times 5 = 7.5$ million cows.

Like Question 1 the first two parts were well answered, although some students could not correctly divide 500 million by 100 in part (a), and 100 million kg was a common answer in (b) (which is actually the amount of methane left after the reduction.) Part (c) proved a bit more difficult, but it was not uncommon for full marks to be awarded in this question.

Question 3: 20 marks (All Years)

Every number is interesting in some way. In this question we will look at the interesting number 41. Firstly, it is a prime number, meaning that it has only two factors, 1 and 41.

(a) 41 is the sum of two *perfect squares*. Find them. (A perfect square is the square of an integer, such as $8^2 = 64$.)

$41 = 4^2 + 5^2 = 16 + 25$.

(b) 41 is a *twin prime*, separated from another prime number by exactly 2. Find the prime number which is a twin to 41.

The two candidates are 39 and 43, but $39 = 13 \times 3$, so the answer is 43.

(c) 41 can be written as the sum of three different prime numbers in a variety of ways. Find one such way.

Any one of the following will work: $3 + 7 + 31$, $5 + 7 + 29$, $5 + 13 + 23$, $5 + 17 + 19$, $7 + 11 + 23$, and $11 + 13 + 17$.

(d) 41 is the sum of exactly six different prime numbers. Find them.

It turns out that the sum of the smallest 6 primes (2, 3, 5, 7, 11, and 13) is 41.

(e) 41 can be found using the formula $[(2n-1)^2+1]/2$, where n is a natural number. Find the value of n that gives 41 using this formula.

Here $n = 5$:

$$\frac{(2\times 5-1)^2+1}{2} = \frac{9^2+1}{2} = \frac{82}{2} = 41.$$

(f) A prime p is a *Sophie Germain prime* if the number $2p + 1$ is also prime. Thus if 41 is a Sophie Germain prime then $2 \times 41 + 1 = 83$ must also be prime. Is 83 prime? If it is, then say 'yes'. If it is not say 'no' and write down the factors of 83.

Yes, 83 is prime. (2, 3, 5, and 7 are not factors, and $11^2 = 121$ means that it can have no other prime factors.)

(g) For many years it was thought that some Mathematical expressions could be used to find prime numbers. For example, the expression $n^2 - n + 41$ (where n is a natural number) gives many primes. So $5^2 - 5 + 41 = 61$, and 61 is prime.

(i) What value is given by the expression $n^2 - n + 41$ if $n = 35$?

$35^2 - 35 + 41 = 1225 - 35 + 41 = 1231$.

(ii) If $n = 41$, then $41^2 - 41 + 41 = 1681$. Is 1681 prime? If it is then say 'yes'. If it is not say 'no' and write down one factor of 1681 (that is not 1 or 1681).

Since $-41 + 41 = 0$, $41^2 - 41 + 41$ is just 41^2, so 1681 is not prime, as it has 41 as a factor.

This 'bookwork' question turned out to be one of the easiest third Questions in the competition for many years, with most students earning a high mark, which was pleasing to see. Common mistakes were seen in part (a) (where students gave 4 and 5 as their answer, when we wanted 16 and 25) and parts (c) and (d) (where 1 and 9 were given as part of the sums; neither are prime). Another less frequent error was in part (e), where n = -4 was given even though it was stated that n was a natural number and therefore positive. There were signs in this question that several students suffered from reading difficulties, restricting them from earning full marks.

Question 4: 20 marks (All Years)

The numbers

$$1,\ 3,\ 6,\ 10,\ 15,\ 21,\ \ldots$$

are called *triangular numbers*.

(a) State the values of the 8^{th} and 9^{th} triangular numbers.

36 and 45.

(b) The triangular numbers can be listed with the natural numbers as a set of ordered pairs (x, y),

$$\{(1,1),\ (2,3),\ (3,6),\ (4,10),\ (5,15),\ (6,21),\ \ldots\}.$$

The formula for the x^{th} triangular number is a quadratic. In other words $y = ax^2 + bx + c$, where a, b, and c are real numbers. It can be shown that $c = 0$.

(i) If $x = 1$, write an equation for a and b.

$y = a \times 1^2 + b \times 1 + 0 = a + b$. Since $x = 1$ means $y = 1$, we have $a + b = 1$.

(ii) If $x = 2$, write an equation for a and b.

$y = a \times 2^2 + b \times 3 + 0 = 4a + 2b$. Since $x = 2$ means $y = 3$, we have $4a + 2b = 3$.

(iii) State the values of a and b.

We solve the equations found in (i) and (ii) simultaneously. We have $a + b = 1$, so $2a + 2b = 2$. Thus $2b = 3 - 4a$ and $2b = 2 - 2a$ gives $3 - 4a = 2 - 2a$, or $1 = 2a$. Hence $a = \frac{1}{2}$, so $b = \frac{1}{2}$ also.

(iv) State the value of the 50th triangular number.

From (iii) above our quadratic is $y = \frac{1}{2}x^2 + \frac{1}{2}x$. If $x = 50$ then $y = \frac{1}{2}50^2 + \frac{1}{2}50 = 1250 + 25 = 1275$.

(c) Show that for any triangular number y, the number $32y + 4$ is always square. **We insert our quadratic equation for y seen above into $32y + 4$:**

$$\begin{aligned} 32y + 4 &= 32(\frac{1}{2}x^2 + \frac{1}{2}x) + 4 \\ &= 16x^2 + 16x + 4 \\ &= 4(4x^2 + 4x + 1) \\ &= 2^2(2x + 1)^2 \end{aligned}$$

Thus $32y + 4$ is a perfect square with roots $\pm 2(2x + 1)$.

In quite a contrast to Question 3, Question 4 proved to be too difficult for the vast majority of students, and was the hardest part of this year's competition. Most students could identify the correct triangular numbers in part (a) (although 28 was too frequently seen, as were 35 and 44), but most of the question requires a reasonable grasp of algebra which most candidates simply did not have. In some cases prior knowledge of triangular numbers may have actually hindered progress through the question. Many students quickly skipped past this question and proceeded to do rather better in subsequent questions, which proved to be good exam technique.

Question 5: 20 marks (All Years)

In a certain school sports team, all students are either 13 or 14 years old (and there is at least one student in each age group). The sum of the ages of all students is 325.

(a) Denote the number of 13-year old students by x and the number of 14-year old students by y. Write down an equation for x and y from the information given.

$13x + 14y = 325$.

(b) What are the prime factors of 325?

Since we can write 325 as $5^2 \times 13$ the prime factors are 5 and 13.

(c) Using (a) and (b), find out how many 13-year old and how many 14-year old students there are in the team, i.e. find the values of x and y. **Rearranging the equation in (a) to $14y = 325 - 13x = 13(25 - x)$ we see that y must be a multiple of 13. Since $y > 0$ and $14 \times 26 > 325$ (which would mean $x \leq 0$) we must have $y = 13$. It follows that $x = 11$, so we have 11 13-year old and 13 14-year old students in the team.**

A lot of students did reasonably well in this question, up to a point. Although many students could find the correct answer in (c), explaining how to reach the values of $y = 13$ and $x = 11$ was not often seen. Furthermore explaining why this was* the only answer *(required for full credit in the question) only occurred a handful of times — while the average mark for Question 5 was higher than that for Question 4, full marks was rather less common.

There were quite a few students capable of producing the correct answer in (c) without producing the correct equation in (a). In (b) quite a few candidates wrote down non-prime factors like 1 and 25.

Question 6: 20 marks (All Years)

Archie brings home a box of cherries for his three grandchildren Joe, Cathy, and Sally, which should be shared fairly between them.

Joe, who was alone at home, is the first to take his portion: he takes one third of the cherries from the box.

Then Cathy comes home. She does not know that Joe already took his cherries, so she takes one third of the remaining cherries.

Finally, Sally takes another third of the remaining cherries.

In the end, 16 cherries are left in the box.

How many cherries did each child take?

Let n be the number of cherries that were initially in the box.

Joe took $\frac{n}{3}$ cherries, so $n - \frac{n}{3} = \frac{2n}{3}$ cherries were left.

Of those, Cathy took $\frac{1}{3} \times \frac{2n}{3} = \frac{2n}{9}$ cherries, so $\frac{2n}{3} - \frac{2n}{9} = \frac{4n}{9}$ cherries were left.

Of these, Sally took $\frac{1}{3} \times \frac{4n}{9} = \frac{4n}{27}$ cherries, so $\frac{4n}{9} - \frac{4n}{27} = \frac{8n}{27}$ cherries were left.

Since these were 16 cherries, we have $\frac{8n}{27} = 16$, and hence $n = \frac{16\times27}{8} = 54$.

Therefore, there were 54 cherries in the box. Of those, Joe took $54/3 = 18$, Cathy took $2\times54/9 = 12$, and Sally took $4\times54/27 = 8$ cherries.

In general most students that attempted this question could find the correct number of cherries that each child took. What took more effort was a convincing explanation about how these numbers were reached. Some students showed no working, others produced reasonably good arithmetical arguments, while the best candidates used an algebraic approach like the one above.

Question 7: 10 marks (Years 10 and 11 only)

A *Pythagorean triple* (a, b, c) consists of three positive integers a, b, c such that $a^2 + b^2 = c^2$.

(a) Find three different examples of Pythagorean triples with $a < b$ and confirm that they satisfy $a^2 + b^2 = c^2$.

Three possible examples are

$$\begin{aligned}(3,4,5) &: 3^2+4^2 = 9+16 = 25 = 5^2,\\ (6,8,10) &: 6^2+8^2 = 36+64 = 100 = 10^2,\\ (9,12,15) &: 9^2+12^2 = 81+144 = 225 = 15^2.\end{aligned}$$

Note that the examples above are all of the form $3n, 4n, 5n$, where $n \geq 1$. Other examples are of the form $(n, \frac{n^2-1}{2}, \frac{n^2+1}{2})$ for any positive odd $n > 1$, such as (5, 12, 13) and (7, 24, 25). Further examples include (8,15,17), (20,21,29), and (12,35,37). It should be clear that there are an infinite number of examples.

(b) Why is there no Pythagorean triple with $a = b$?

If there was a Pythagorean triple with $a = b$, then it would satisfy the equation $2a^2 = c^2$. But this would mean $c = \sqrt{2}a$, which is impossible since c is rational (it is an integer) and $\sqrt{2}$ is irrational.

This was a reasonably straightforward question. In part (a) many students were able to provide 3 triples, but neglected to read the full question and provided no (or insufficient) evidence that they worked.

In part (b) a lot of candidates could reach $2a^2 = c^2$, but were then unable to explain what this meant.

Question 8: 10 marks (Year 11 only)

John locked his bike with a 4-digit combination lock several months ago, and he cannot recall the correct combination. Each digit is one of the numbers 0, 1, 2, . . . , 9.

John only knows that each of the numbers 1, 4, and 6 appears exactly once, but he can't remember the position of those numbers, and he does not know what the fourth number was.

(a) What is the maximum number of combinations he would need to try to open the lock?

We first consider those combinations for which the known numbers are ordered 1, 4, then 6. Let x be the unknown number. Since x is neither 1 nor 4 nor 6, there are 7 possible values for x: (0, 2, 3, 5, 7, 8, 9).

Since x has 4 possible positions (first, last, between 1 and 4, and between 1 and 6), we obtain $4 \times 7 = 28$ possible combinations in which the numbers 1, 4, 6 appear in this order.

The numbers 1, 4, and 6 could appear in other orders as well. There are 6 different orders in total, and each of these 6 orders has 28 possible combinations. Hence, in total there are $6 \times 28 = 168$ combinations that John would need to try in the worst case.

(b) If John knew that the first digit was the 4, how many combinations would he then need to try at most?

If the first number is 4, then the other given numbers 1 and 6 could appear in two orders.

Moreover, there are again 7 possible values for the remaining number x, and x could be in 3 different positions (second, third, or last). Together, there are $2 \times 7 \times 3 = 42$ possible combinations.

This question was reasonably well done by most of the Year 11 students that attempted it. Almost everyone who got (a) correct got (b) correct as well, with marks only being lost for a lack of explanation in both parts. The most common mistake was not to eliminate 1, 4, and 6 from the possible values of x.