Year 9 (Form 3) Prize Winners

First	Grace Wu	St Cuthbert's College
Second	Bu Fan Zhao	ACG Parnell College
Third	Tanupat Trakulthongchai	Auckland Grammar School

Top 30 (in School Order):

Luke Kowalski, ACG Parnell College
Shanying Liu, ACG Parnell College
Xiaotian Xu , ACG Parnell College
Oliver Gunson, Auckland Grammar School
Thomas Harnett, Auckland Grammar School
Ethan Moy, Auckland Grammar School
Roshan ter Wal, Auckland Grammar School
Lachlan Trotman, Auckland Grammar School
Annabelle Brownsword, Burnside High School
David Evans, Canterbury Home Educators
Clara Dujakovic, Cashmere High School
Dhon Lao, Liston College
Ryan Dukeson, Macleans College
Gina Kim, Macleans College
Michael Ma, Pinehurst School
JESSICA MENG, Rangitoto College
Alanna Santoso, Sancta Maria College
Vicky Ngo, Selwyn College
Gemma Lewis, St Andrew's College
Jifei Shao, St Cuthbert's College
Ena Yin, St Cuthbert's College
Eric Lee, St Kentigern College
Eric Liang, St Kentigern College
Emma Ying, St Kentigern College
Scott (Tianxiao) Zhou, St Paul's Collegiate School
Joe McKibbin, Takapuna Grammar School
Jed Gouldson, Tauranga Boys' College

Year 10 (Form 4) Prize Winners

First	James Xu	Kristin School
Second	Andrew Yu	Kings College
Third	Ryaan Sidhu	Auckland Grammar School

Top 30 (in School Order):

Daniel Johnston, Auckland Grammar School
Aaron Wang, Auckland Grammar School
Kelvin Xiao, Auckland Grammar School
ZOE SMITH, Avondale College
Haylie Wong, Diocesan School for Girls
Janet Guo, Hillcrest High School
Hayley Sharpe, Hillcrest High School
Martin Brook, John McGlashan College
Jacky Zhang, Liston College
Eason Chang, Macleans College
James Hui, Macleans College
Alvari Kupari, Macleans College
Jack Liu, Macleans College
Gary Ni, Macleans College
Hosea Tong-Ho, Macleans College
Bob Xiang, Macleans College
Frank Yau, Macleans College
SUN WOONG KANG, Rangitoto College
ELLA LIN, Rangitoto College
Terry Qi, Scots College
Tom Edwards, St Andrew's College
Corin Simcock, St Andrew's College
Amy Prebble, St Kentigern College
Joanna Li, St Peter's School (Cambridge)
Lachlan Jardine, Takapuna Grammar School
JiYong Kim, Tauranga Boys' College
Susanna Weng, Wellington Girls' College

Year 11 (Form 5) Prize Winners

First	Phillip Han	Auckland Grammar School
Second	Rick Han	Macleans College
Third	Oliver Dai	Macleans College

Top 30 (in School Order):

Jenna Kayleigh Parkin, ACG Parnell College
Zelin Shao, ACG Parnell College
Samuel Blyth, Auckland Grammar School
Hangyeol Kim, Auckland Grammar School
GAYATRI GANESH, Avondale College
Nicholas Grace, Burnside High School
Aditi Sharma, Burnside High School
Shanika Yu, Burnside High School
Evan French, Glendowie College
Nick Ni, James Hargest College (Senior Campus)
Tobias Devereux, Kavanagh College
Jason Lee, Kings College
Amy Chen, Kristin School
Sophia Fang, Kristin School
Eric Zhang, Kristin School
Megan Macdiarmid, Logan Park High School
Nicholas Lianto, Macleans College
Isaac Liu, Macleans College
Zenobia Lu, Macleans College
Ellen Wang, Macleans College
Kris Zhang, Macleans College
Melissa Zhang, Macleans College
Jonnie Moffett, Napier Boys' High School
Abel McNabb, Nelson College
Siddhartha Gurung, Otumoetai College
CAMERON SENIOR, Rangitoto College
Dominic Baron, St Peter's College (Epsom)

2019 Model Answers

As always only one method is shown. Often several methods exist. We don't guarantee that the method shown is the best or fastest.

Question 1: 10 marks (Years 9 and below only)

(a) Write A or B for the correct answer. You do not need to show working. **In general part (a) was well answered.**

(i) Tea costs $NZ3.29 a packet. Baked beans cost $NZ1.59 a can. Which is cheaper? (A) 1 packet of tea and 4 cans of baked beans, *or* (B) 2 packets of tea and 3 cans of baked beans. **A ($9.65 compared to $11.35).**

(ii) Which is the cheaper deal per 100g? (A) 400g of cheese at $NZ4.99, *or* (B) 1 kg of the same cheese at $NZ11.49. **B (roughly $1.15 compared to roughly $1.25).**

(iii) On a particular day the exchange rate for the United States dollar is given by: $US1 = $NZ0.696. What is $NZ500 worth in $US to the nearest dollar? (A) $US348, *or* (B) $US718. **B. According to the question the $US is worth less than the $NZ so there must be more $US for $NZ500. This was a mistake on our part (the $US is actually worth more) but within the competition the correct answer had to be B.**

(b) If 3kg of flour costs $NZ9.60 how much does it cost to buy 7kg in $NZ? **$22.40. Divide 9.6 by 3 and multiply by 7. Generally well answered although many candidates gave the "calculator" answer $22.4 and received reduced credit.**

(c) Anja borrows $NZ400 at 15% interest compounded annually at the start of each new year (there is no interest during the first year). She doesn't pay anything back during the first two years.

(i) How much does Anja owe altogether after the first amount of interest is added on? **$460. Well answered, although a few students answered $60 and forgot that Anja still owed the $400 as well.**

(ii) How much does she owe altogether after the second amount of interest is added on? **$460 × 1.15 = $529. Fairly well answered, but applying a second interest charge was well beyond some.**

(iii) During the third year Anja pays back $NZ400. How much does she still owe after the third amount of interest is added on?

$$\$529 - \$400 = \$129.$$
$$\$129 \times 1.15 = \$148.35.$$

Partial working credit was applied to those who obtained the wrong answer but clearly subtracted $400.

Question 2: 10 marks (Years 10 and below only)

Consider the number 854.

(a) Reverse the digits and find the difference between 854 and the number you found, taking the smaller number away from the larger one. **$854 - 458 = 396$. Meant to be easy, but some students clearly wrote the subtraction down then couldn't obtain the answer. This is worrying! A few subtracted the larger number from the smaller, obtaining an incorrect negative answer.**

(b) Reverse the digits of the answer to (a) and add this number to the answer in (a). What result do you get? **1089. (396 + 693 = 1089.) It was depressing to see incorrect answers!**

(c) Repeat steps (a) and (b) with any three digit number *abc* of your your own choosing where $a > c + 1$. Do you get the same result as you found in part (b)? **Following the instructions does give the same answer. Some students don't understand what ">" means. Many did obtain 1089 but didn't say "Yes" or equivalent. If we ask a question we expect an answer.**

(d) The number 854 can be written in expanded form as $8 \times 100 + 5 \times 10 + 4$. Write the general 3-digit number *abc* (where $a, b, c > 0$ and $a > c$) in expanded form. **$100a + 10b + c$. Commonly a student picked a number and worked with that. No credit resulted. This question seemed to divide students into two groups — those who did well and those who didn't. There were exceptions.**

(e) Now consider the "reversed" number *cba* (where $a, b, c > 0$ and $a > c$). Write that in expanded form and subtract it away from the expanded form of *abc*, simplifying your answer as far as possible. **$99a - 99c$ or $99(a - c)$.**
There was partial credit for writing $100c + 10b + a$. Then the algebraic forms had to be successfully subtracted. Brackets around the $(100c + 10b + a)$ are essential.

Question 3: 20 marks (All Years)

This question mainly involved "finding a pattern". It was fairly "easy" for many students. Full credit was common. For example several students below Year 9 level completely mastered the question.

(a) Find the sum of the arithmetic series T = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10. (An arithmetic series has a constant difference between consecutive terms.) **55. Well answered, although 45 was not uncommon (from students who forgot to add the "10").**

(b) If $a = 1$ and $z = 10$ are the first and last terms of the series T in (a) then $a + z = 11$. How many pairs of numbers altogether (including 1 and 10) in the series T add to 11? **Five. (1 + 10, 2 + 9, 3 + 8, 4 + 7, 5 + 6.) Well answered, although a few candidates missed a pair and "10" was seen from students who thought they should reverse all the pairs.**

(c) Write the answer to (a) as the product of two whole numbers. **5 × 11 (or 11 × 5). The rare answer 1 × 55 was awarded partial credit. Several students did not know what the word "product" meant and they found a pair which summed to 55.**

(d) With brief working, hence or otherwise find the sum of the series U = 1 + 2 + 3 + . . . + 98 + 99 + 100. **Here students had to recognise the pattern. There are 50 pairs each adding to 101 and so 50 × 101 = 5050. Although this was well answered a few candidates tried to add all the terms. Remarkedly one or two reached the correct answer! However for some reason they did not complete many questions.**

(e) True or false? (You do not have to show working.)

(i) The first and last terms of the series V = 1 + 2 + 3 + . . . + 48 + 49 + 50 add to 51. **True. Gift marks, but occasionally missed out.**

(ii) The sum of the series V = 1 + 2 + 3 + . . . + 48 + 49 + 50 is 1275. **True. There are 25 pairs summing to 51.**

(iii) There are 501 pairs of numbers in the series W = 1 + 2 + 3 + . . . + 998 + 999 + 1000 that add to 1001. **False. There are only 500.**

(iv) The sum of the series W = 1 + 2 + 3 + . . . + 998 + 999 + 1000 is 500 500. **True. 1001 × 500. This was also "easy" but for some reason several candidates missed it out completely and moved straight onto part (f).**

Now consider the general arithmetic series S = $a + \ldots + z$ which has exactly n terms.

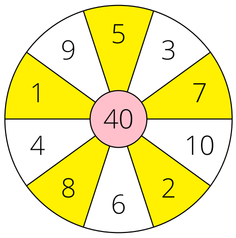
(f) In terms of n how many pairs of terms of the series S add to $(a + z)$? **$n/2$ or equivalent. The incorrect answer 13 (half the 26 letters in the alphabet) was common from those yet to understand Algebra fully.**

(g) Write down in terms of n, a, and z your best guess (based on this question) for the formula of the sum of n terms of the series S.

$$\frac{n}{2}(a + z).$$

If there were to be problems in this question (apart from missing parts out) it would be in (f) and (g). Algebra (as expected) is a bridge too far for many. The answer $\frac{n}{2}(2a + (n - 1)d)$ had nothing to do with the question and earned no credit. The question does say "based on this question".

Question 4: 20 marks (All Years)



Mini-darts is a darts game for two junior players. The dartboard has 10 equal segments, numbered 1 to 10. If a dart lands in a segment the player earns the points for that segment. A central bulls-eye is worth 40, but missing the board earns 0. After tossing a coin to see who throws first each player in turn may throw up to three darts per visit to the board. They can land all three darts in the same scoring zone. The winner is the first to reach exactly 261 from as many visits to the dartboard as needed. If they go over 261 they lose all their points from that visit to the table and must sit down immediately.

(a) True or false? (You don't need to show working.) **These were generally well done.**

(i) The most you can score with three darts is 120. **True. 3 × 40.**

(ii) You can reach 261 in a minimum of three visits to the mini-dart board. (One visit is up to three throws of the darts at the board.) **True. For example 120 and 120 and 21.**

(iii) If you need 18 to win you can do it with two darts but one dart is not enough. **True. Some people thought that 40 (only one dart) would suffice. They failed to properly read that if you go over 261 you lose all points from that round.**

(b) Clint has three darts with which to score exactly 3 to win. What is the minimum number of throws he needs to win? **One. A gift.**

(c) In another game Clint has three darts to score exactly 21 to win. What is the minimum number of throws he needs to win? **Three. Another gift.**

(d) In a third game Clint has three throws to score exactly seven (7) to win.

(i) If he misses the board completely with the first dart but reaches a score of seven with exactly two more throws in how many different ways can he throw the two darts without worrying about order? **Four. They are (in any order) (4, 3), (5, 2), (6, 1), and (0, 7). People who didn't think about (0, 7) received partial credit.**

(ii) If Clint throws seven to win in exactly two throws (and so doesn't need the third dart) list as ordered pairs all the ways he could do this. For example (4, 3) is one way. This time the order of the throws is counted. **(4, 3), (3, 4), (5, 2), (2, 5), (6, 1), (1, 6), and (0, 7). Partial credit was allowed for six ways if (0 , 7) was missed. The "answer" (7, 0) was not allowed because it only needs one throw, not the required two. It wouldn't make sense to throw a "7" then deliberately miss the board because after throwing the "7" the game is already over.**

(iii) If Clint needs all three darts (and none of them scores zero) list as triples all the ways he could do this. For example, (5, 1, 1) and (4, 1, 2) and (3, 2, 2) are ways Clint could score 7 points with three darts.

(5, 1, 1),	**(1, 5, 1),**	**(1, 1, 5),**
(4, 2, 1),	**(4, 1, 2),**	**(2, 4, 1),**
(2, 1, 4),	**(1, 4, 2),**	**(1, 2, 4),**
(3, 2, 2),	**(2, 3, 2),**	**(2, 2, 3),**
(3, 3, 1),	**(3, 1, 3),**	**(1, 3, 3).**

(15 in total, the placement of each triple in the list doesn't matter.) We wanted order to count. The word "all" implies that. Partial credit was awarded to those who gave the triples where order doesn't count or stated "15" without a complete list.

Question 5: 20 marks (All Years)

Many scientists believe that the Earth was formed about 4.5 billion years ago, where one billion is equal to 1000 million (and where one million = 1 000 000). In this question we will assume that this is correct. If the age of the earth were to be compressed into 24 hours then each hour would represent 187.5 million years in real time.

(a) Using this scale how many real time years would 12 compressed hours represent? **2.25 billion (years) or equivalent. Surprisingly there were many mistakes. 12 hours is half a day (so divide 4.5 by 2) but there's so much to do that some students didn't read the question carefully.**

(b) Using this scale how many real time years would 20 compressed minutes represent? **62.5 million (years) or equivalent. 20 minutes is one third of an hour (divide 187.5 by 3). Poor reading occured again.**

(c) Explain mathematically in one line only how we worked out the number 187.5 million. **Divide by 24 (well done) and multiply by 1000 (or something equivalent) to convert billions into millions. This second operation was often missed.**

(d) If 4.5 billion = 4.5×10^9 then write 4.5 billion as an "ordinary" number. **4 500 000 000. Well answered. The easiest part of the question.**

(e) If dinosaurs became extinct approximately 66 million years ago in real time, at what time on the 24 hour clock did dinosaurs become extinct on Earth? Give your answer to the nearest half-hour. **Here is one method (there are many):**

$$66\,000\,000/4\,500\,000\,000 = 0.014\,666\ldots$$
$$1 - 0.014\,666 = 0.985\,333\ldots$$
$$0.985\,333 \times 24 = 23.6\ldots$$

About 11:30 at night or 23:30. Here the "large" numbers (billions) really show their effect. We think that dinosaurs became extinct a very long time ago (66 million years). But on a compressed 24 hour scale it's less than half an hour ago.

(f) If modern humans have been in existence for 200 000 years in real time at what time on the 24 hour clock did modern humans first exist on Earth? Give your answer to the nearest minute.

$$200\,000/4.5 \text{ billion} = 0.000\,044\,44\ldots$$

There are 24 × 60 = 1440 minutes in a day.

$$0.000\,044\,44\ldots \times 1440 = 0.064 \text{ minutes (or about 4 seconds).}$$

Humans have existed for 4 seconds on the 24 hour clock. Surprisingly enough (perhaps) this shows that modern humans have been on the earth since 12:00 midnight or 24:00 to the nearest minute.

(g) If life began on Earth at 5 am in the morning on the 24 hour clock we are using in this question how many years ago in the real time life of the Earth did life begin?
4.5 billion × 19/24 = 3 562 500 000 (years ago).
Multiplying by 5/24 earned partial credit.

Question 6: 20 marks (All Years)

A prime number has exactly two factors (1 and itself). The number 1 (which is not prime) only has one factor (namely itself), while composite numbers (all numbers that are not 1 and are not prime) have at least three factors. For example, 103 is prime, with factors 1 and 103. 30 has 1, 2, 3, 5, 6, 10, 15, and 30 as factors (so eight factors in total), while 25 has only three factors (namely 1, 5, and 25). **This question proved to be the most difficult to complete, although there was "easy" credit, especially at the start. Only one candidate, the runner up at Year 11 level, managed full marks in Question 6.**

(a) Find the smallest three numbers with exactly 2, 3, and 4 factors respectively. **2, 4, 6. 2 has 2 factors (1 and itself). 4 has 3 factors (1, 2, and 4). 6 has 4 factors (1, 2, 3, and 6).**

(b) Briefly explaining your answer (perhaps with a list), how many numbers less than 20 have

(i) 2 factors? **We want primes less than 20. The list is 2, 3, 5, 7, 11, 13, 17, and 19. So there are eight numbers in total. It was noticeable that many candidates listed all the numbers but failed to say how many there were. This automatically reduced the credit.**

(ii) 3 factors? **We want perfect squares of primes here. So 4 and 9. Two numbers in total. (16 has 1, 2, 4, 8, and 16 as factors, so there are more than three factors.) Note that the first two parts were well answered but many students couldn't sort out the next two.**

(iii) 4 factors? **Here numbers that are the product of two primes qualify. So 6, 10, 14, and 15. 8 also qualifies as one of its factors is the square of a prime. Five numbers in total. It was noticeable that 15 was often missed out.**

(iv) more than 4 factors? **This is all numbers not listed above, excluding 1 (which has only one factor) and 20 itself (the question asked for numbers less than 20). So 12, 16 (see above), and 18. Three numbers in total.**

(c) Briefly explain why the only numbers with exactly 3 factors have the form p^2, where p is a prime number. **Numbers of the form p^2 have the factors 1, p, and p^2. Prime numbers have exactly two factors, and 1 has one factor. If $x \neq p^2$ is not prime or 1, it is composite. There are two cases. If $x = p^n s$ where p is prime and $n > 2$ then x has at minimum the factors 1, p, p^2, and p^3 (so more than three factors regardless of what s is). Otherwise x is square-free, so $x = qrs$ where q and r are distinct primes. Then x has at least 1, q, r, and x as factors. (If $s = 1$ these are the only factors.) Hence x has at least four factors.**

(d) Can a number have exactly 5 factors? Briefly explain your answer, giving an example less than 100 if your answer is yes. **Yes. For example 16 has five factors. In fact any number of the form p^4 has five factors: 1, p, p^2, p^3, and p^4. Another example less than 100 is $3^4 = 81$.**

(e) A square-free number does not have any factors of the form p^2, where p is a prime number. Explain why every square-free number other than 1 has an even number of factors. **Prime numbers have two factors. So consider a composite number x of the form qy where q is prime, $y \neq 1$, and q is not a factor of y (otherwise x would not be square-free). For each factor s of y, there exists two equivalent factors of x, namely s and qs. Thus if y has n factors, x must have $2n$ factors.**

Question 7: 10 marks (Years 10 and 11 only)

Fahu has 12 identical (except for colour) cards: four red, four yellow, and four blue. He shuffles and draws three cards at random. Only one of three things can happen:

- Case 1: They are all the same colour.
- Case 2: They are all different colours.
- Case 3: Two are of one colour and one is of another.

Which of the three cases is the most likely? Show all necessary working. If your answer is correct but you do not show working you will receive no credit.
Case 1:

$$\begin{aligned}\textbf{Probability(3 Red)} &= \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10} \\ &= \frac{1}{55}.\end{aligned}$$

But there are three colours in total so Case 1's probability is 3/55.
Case 2:

$$\begin{aligned}\textbf{Probability(Red then Yellow then Blue)} &= \frac{4}{12} \times \frac{4}{11} \times \frac{4}{10} \\ &= \frac{8}{165}.\end{aligned}$$

But (all different) can occur in exactly six ways: (R, Y, B), (R, B, Y), (Y, R, B), (Y, B, R), (B, Y, R), and (B, R, Y).

$$\begin{aligned}\textbf{So probability(all different)} &= 6 \times \frac{8}{135} \\ &= \frac{16}{55} \textbf{ (when simplified).}\end{aligned}$$

But the three Cases are the only possible outcomes. The probabilities must sum to 1. We can find Case 3's probability by subtraction. Case 3:

$$\textbf{Probability(2 of one colour and one of the other)} = 1 - (\frac{3}{55} + \frac{16}{55}) = \frac{36}{55}.$$

So Case 3 is the most likely.

Question 7 was generally poorly handled. The answer 1/55 was often found but most people failed to go further. At least the people who reached something over 55 (or equivalent) knew to adjust probabilities as they went. Several candidates incorrectly worked out $1/3 \times 1/3 \times 1/3 = 1/27$ which meant they didn't realise that if they drew, say, a red first then the probabilities of later draws had to be adjusted. Also some candidates worked out Cases 1 and 2 correctly but then failed to subtract the sum of these two away from 1. Having said all that there were individuals who answered the question perfectly in four lines.

Question 8: 10 marks (Year 11 only)

Barbara the Builder has to build a backyard swimming pool for a rich client. There are four requirements:

- It must be rectangular.
- The depth must be 1.5 m.
- The surface area of the pool must be 275 m^2.
- The length of a diagonal (from corner to opposite corner) must be 25 m.

If x is the breadth of the pool and y is the length, requirements 3 and 4 lead to the two equations:

- equation A: $xy = 275$.
- equation B: $x^2 + y^2 = 625$.

(a) Make y the subject of equation A.
$y = 275/x$. Practically the only correct answer in the question for many.

(b) Solve the equations simultaneously. Determine every real solution even if an answer contains negative results, giving your answers as ordered pairs.

$$x^2 + (275/x)^2 = 625$$ **Several candidates stopped here.**

$$x^4 - 625x^2 + 75625 = 0$$ **And several stopped here when they realised they had a quartic.**

$$x^2 = \frac{625 \pm \sqrt{390625 - 302500}}{2}$$

The quadratic formula was provided on the front page. A few students performed a substitution first, putting $x^2 = z$ (say) and later on substituting back to x.

$$x^2 = \frac{625 \pm 296.859}{2}$$

$$x^2 = 460.929 \text{ or } 164.071$$

Therefore $x = \pm 21.469, \pm 12.809$ (3 d.p.) **Four correct solutions for x.**

Thus $y = \pm 12.8, \pm 21.5$, so there are four solutions for y. Potentially there are eight ordered pair solutions in (x, y). But equation A implies that x and y must have the same sign. So there are only four ordered pair solutions altogether:

(21.5, 12.8), (12.8, 21.5), (-21.5, -12.8), (-12.8, -21.5).

(c) Give the dimensions of the feasible solution to Barbara's problem if x is less than y. Round your answers to one decimal place if necessary.
The realistic dimensions (with $x < y$) are breadth (x) = 12.8 m, length (y) = 21.5 m and depth = 1.5 m. Many students who got this far forgot about the depth (but it would have been "extra" information if it hadn't been used somewhere).