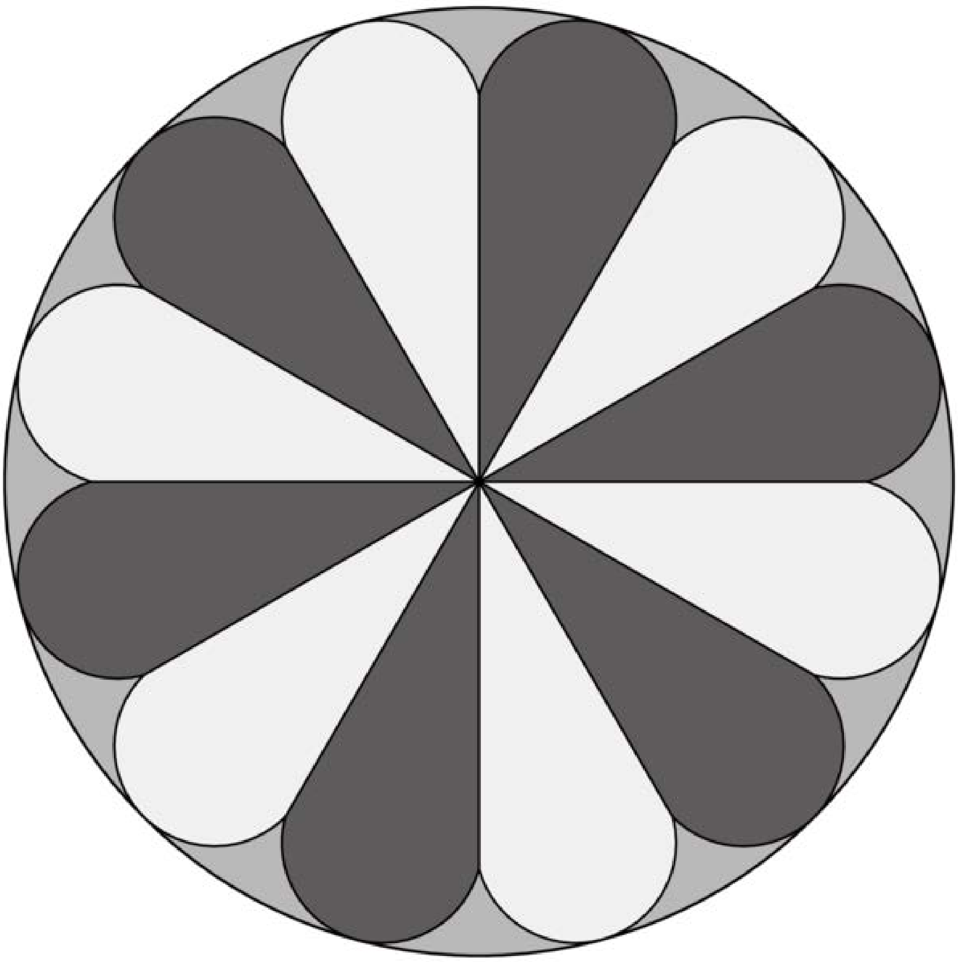
Department of Mathematics and Statistics

Junior Mathematics Competition 2017 Solutions and Comments

Web: maths.otago.ac.nz/jmc **Twitter:** @uojmc



Year 9 (Form 3) Prize Winners

First	Tobias Devereux	Kavanagh College
Second	Ethan Ng	Merrin School
Third	David Lu	Pinehurst School

Top 30 (in School Order):

Samuel Blyth, Auckland Grammar School
Louie Lu, Burnside High School
Jasmine Ha, Diocesan School for Girls
Sam Zhuang, Kristin School
Shirley Bian, Kristin School
Oliver Dai, Macleans College
Abhinav Chawla, Macleans College
Sophia Liu, Macleans College
James Seo, Otago Boys' High School
Reina Zhang, Rangitoto College
James Sutton, Rongotai College
Koa Yoshihara, St Kentigern College
Andy Wu, Takapuna Grammar School
JOO-YEON JEONG, Avondale College
Reuben Clarkson, Cashmere High School
Jason Lee, Kings College
Christian Newey, Kristin School
Eric Zhang, Kristin School
Kris Zhang, Macleans College
Andrew Zeng, Macleans College
Rick Han, Macleans College
Angelina Del Favero, Queen Margaret College
Ethan Lu, Rangitoto College
Michelle Guan, St Kentigern College
Haokun He, St Kentigern College
Thomas Bailey, Waimea College
Danyi Zhao, Westlake Girls' High School

Year 10 (Form 4) Prize Winners

First	Grace Chang	St Kentigern College
Second	Ming Wang	St Cuthbert's College
Third	Mackinley He	Auckland Grammar School

Top 30 (in School Order):

Luke Bao, Auckland Grammar School
Isaac Yuan, Auckland Grammar School
Nathan Chen, Auckland Grammar School
Ethan Qi, Glendowie College
Kunli Zhang, Kristin School
Cathy Zeng, Logan Park High School
Angela Yang, Macleans College
Terry Shen, Macleans College
Ranudi Lewlwala, Macleans College
Boa Kim, Macleans College
Megan Tse, Pakuranga College
Ciaran Carroll, Palmerston North Boys' High School
Carlie Yung, St Cuthbert's College
Daniel Gong, Auckland Grammar School
Qianhao Luo, Auckland Grammar School
Jordan Peters, Burnside High School
George Bates, King's High School
Vanessa Xiong, Kristin School
Dillon Hong, Macleans College
Darsh Chaudhari, Macleans College
Jimmy Zhou, Macleans College
Jasmine Zhang, Macleans College
Young Yu, Macleans College
Cameron Van Rynbach, Palmerston North Boys' High School
Minju Kim, Rangitoto College
Belinda Hu, St Cuthbert's College
Heeju Rho, St Kentigern College

Year 11 (Form 5) Prize Winners

First	Sonia Shao	Kristin School
Second Equal	Marcus Ooi	Kings College
Second Equal	James Mead	Kings College

Top 30 (in School Order):

Serena Jou, ACG Strathallan College
Tianyu Chi, Auckland Grammar School
Liam Wong, Auckland Grammar School
Louie Wei, Auckland International College
Jing Qu, Botany Downs Secondary College
Mukund Karthik, Botany Downs Secondary College
Eric Song, Burnside High School
Felix Backhouse, Burnside High School
Daniel Chong, Christchurch Boys' High School
Callum Sng, Kings College
Chris Brand, Kristin School
William Han, Macleans College
Jian Pan, St Cuthbert's College
Owen Sun, Auckland Grammar School
Yiren Zhu, Auckland Grammar School
Seung Jae Hwang, Auckland International College
Soohyun Kim, Auckland International College
Daniel Liang, Botany Downs Secondary College
Yash Shahri, Botany Downs Secondary College
Andrew Chen, Burnside High School
Josephine Situ, Carmel College
Paddy Borthwick, King's High School
Edward Day, Kings College
Nicholas Yao, Macleans College
Songyan Teng, Pakuranga College
Alice Cao, St Paul's Collegiate School
Ethan Gray, St Peter's College (Epsom)

Question 1 (Years 9 and below only)

(a) Over a five day period at Kakanui during April the daily amount of rainfall was recorded.

Day:	1	2	3	4	5
Rainfall (mm):	3	0	4	26	5

(i) How much total rainfall was there over the five day period? 38 mm. Well done. But see below.

(ii) Show that the average rainfall per day over the five day period was 7.6 mm. 38/5. Well done.

(b) Over the same month (30 days) the average rainfall per day at Kakanui was 5.6 mm.

(i) What was the total rainfall for April? $5.6 \times 30 = 168$ mm. Well done.

(ii) What was the average rainfall for the 25 days not included in the table in part (a)? $168 - 38 = 130$, so $130/25 = 5.2$ mm. Pleasing. A good number of students could handle the 'adjustment'.

(c) On Day 2 the maximum temperature was 16°C (degrees Celsius). On Day 3 the maximum temperature was 10% less than that on Day 2. What was the maximum temperature on Day 3? 14.4°C. Well done.

(d) Some countries measure temperature using the Fahrenheit scale, while New Zealand uses Celsius. The formula for converting Celsius (C) to Fahrenheit (F) is $F = \frac{9}{5}C + 32$.

(i) If the temperature in degrees Celsius is 20° what is the temperature in degrees Fahrenheit? $F = 9/5 \times 20 + 32$, so $F = 68°$. Well done.

(ii) If the temperature in degrees Fahrenheit is 50° what is the temperature in degrees Celsius? $50 = 9/5C + 32$, so $18 = 9/5C$, hence $C = 10°$. Fairly well done.

(iii) Find the temperature where $F = C$. $F = 9/5F + 32$, which gives $-4/5F = 32$, and so $F = -40°$. Not well answered. It was rare to see pupils write x = 9/5 x + 32 and then solve this equation. The correct answer was more often achieved via guess and check.

Question 2 (All Years)

A prime number has exactly two factors, 1 and itself. By this definition, 1 is not a prime number. The first fifteen prime numbers are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$$

To check whether a number is prime or not, one method is:

(1) Use a calculator to find the square root of the number (for example $\sqrt{41} = 6.403$ to three decimal places).

(2) Divide the number (x say) by all the prime numbers less than $\sqrt{x}$ in decreasing order. For example for 41 you have to divide 41 by 5, divide 41 by 3, then divide 41 by 2. If it is divisible by any of these numbers then it is not a prime. However, you do not have to divide by 7 (or anything higher).

With this information answer the following questions:

(a) Germain primes are named after the great French mathematician Sophie Germain. A prime p is a Germain prime if the number $2p + 1$ is also a prime. For example, 41 is a Germain prime because $2 \times 41 + 1 = 83$, which is also prime.
Are the following statements true or false?

(i) 3 is a Germain prime. True. $2 \times 3 + 1 = 7$. Also prime. Well answered.

(ii) 11 is a Germain prime. True. $2 \times 11 + 1 = 23$. Also prime. Well answered.

(iii) 23 is a Germain prime. True. $2 \times 23 + 1 = 47$. Also prime. Well answered.

(iv) 2017 is a Germain prime. False. $2 \times 2017 + 1 = 4035$. Clearly divisible by 5. Well answered although some checked by dividing by every possible prime before they saw the 'obvious' answer. (4035 also has 3 and 269 as prime factors.)

(b) 2017 is prime. Show that 2009 is not prime. Hint: Note that $\sqrt{2009} = 44.82$ to 2 decimal places. 2009/7 (or 2009/41) is a whole number (287 and 49 respectively). Often well answered but many students divided by 2, 3, and 5, stopping there and stating that 2009 must be prime.

(c) There is only one prime between 2006 and 2016. State its value. You do not have to show working. After eliminating all the even numbers along with 2007, 2013 (digits add to a multiple of 3 so they must be divisible by 3), 2009 (see above), and 2015 (obviously divisible by 5), the only number left is 2011. Well answered but many students did all the divisions on their calculator then wasted time by writing them all down.

(d) We define whole numbers a and b to be *cousin primes* if a and b are primes and $a - b = 4$. For example 3 and 7 are cousin primes because 7 - 3 = 4.

(i) Find a pair of cousin primes where both are between 10 and 20. 13 and 17. Well answered.

(ii) Find a pair of cousin primes where both are between 100 and 120. There are two possible answers: 103 and 107 or 109 and 113. Some people chose 113 and 117, but the digits of 117 add to 9, so it must be divisible by 3.

(iii) Is 2017 one of a pair of cousin primes? Show your reasoning. No. 2013 is divisible by 3 and 2021 is divisible by 43. Fairly easy but not so well answered. Many people stated that 2021 was prime. In too many cases 2013 was given as prime even though in (c) the correct answer of 2011 was given.

Question 3 (All Years)

In one episode of The Simpsons, the crowd at a baseball game was asked to guess the attendance. There were three options: 8128, 8191, and 8208. All three numbers are interesting from a mathematical point of view.

(a) The first, 8128, is a perfect number. This means that after you find all the factors of it and add them up (apart from 8128 itself), the result is 8128. The first 'perfect' number is 6, because when you add the factors of 6 (apart from 6 itself) you get 1 + 2 + 3 = 6.

(i) Write down in ascending order all the factors of 28 (apart from 28 itself), including 1. 1, 2, 4, 7, 14. Well answered but some included 28 despite being told not to.

(ii) Is 28 a perfect number? (You do not have to show working.) Yes. Well answered.

(iii) Write down in ascending order all the factors of 8128 (apart from 8128 itself) including 1. (Hints: there are 13 factors, not counting 8128 itself. One of them is 64.) 1, 2, 4, 8, 16, 32, 64, 127, 254, 508, 1016, 2032, 4064. Well answered. We did not penalise the many students who wrote their answer in descending order. Knowing the meaning of the word 'ascending' was not essential. Strangely enough fewer students wrote 8128 here than wrote 28 in part (a).

(b) The second number, 8191, is a Mersenne prime number. A prime number has exactly two factors, 1 and itself, and a Mersenne prime has the form

$$2^p - 1,$$

where p is a prime number. For example, if $p = 3$, the number $2^3 - 1 = 7$ is a Mersenne prime. Not all numbers of the form $2^p - 1$ are prime. For example, if $p = 23$ then $2^{23} - 1 = 8388607$, which is not prime since $8388607 = 47 \times 178481$.

(i) Solve for p the equation

$$2^p - 1 = 8191.$$

You do not need to show working. $p = 13$. Well answered. Either students wrote out 2×2 13 times or they used logs:

$$\begin{aligned} 2^p - 1 &= 8191 \\ 2^p &= 8192 \\ \log 2^p &= \log 8192 \\ p \log 2 &= \log 8192 \\ p &= 13. \end{aligned}$$

(ii) Show that $2^{11} - 1$ is not a prime number. $2^{11} - 1 = 2047$. 2047 is divisible by 23 (and 89). Many people obtained 2047 (although some reached 21 or 2049). It was more troublesome to find the factors.

(c) The third number, 8208, is a narcissistic number. There are four digits and when you add their fourth powers it makes 8208. In other words

$$8^4 + 2^4 + 0^4 + 8^4 = 8208.$$

Note that if there are two digits the power involved is 2. If there are three digits the power involved is three. So a three digit example is $370 = 3^3 + 7^3 + 0^3$.

(i) Show whether 35 is narcissistic or not. $3^2 + 5^2 = 34$. No. Well answered, although some used 4^{th} powers (they obviously did not read the full question).

(ii) Show whether 153 is narcissistic or not. $1^3 + 5^3 + 3^3 = 153$. Yes. Similar to the previous question.

(iii) Explain why there are no two digit narcissistic numbers starting with 1. Students found several methods but many completely missed this out:

- $5^2 = 25$. This means our second digit must be smaller than 5. But $1^2 + 0^2 = 1$. Also $1^2 + 1^2 = 2$, $1^2 + 2^2 = 5$, $1^2 + 3^2 = 10$, and $1^2 + 4^2 = 17$. None are narcissistic.
- 'Brute force', working every option out.
- Since 1^2 is odd, adding it to an odd number must make the result even (and vice-versa). Since the square of an odd number is odd and the square of an even number is even, no number starting with 1 can be narcissistic.
- (the most 'mathematical' way). We must have $x^2 + 1^2 = 10 + x$. This simplifies to $x^2 - x - 9 = 0$. The discriminant is 37 which is not a perfect square, so there are no integer solutions.

Question 4 (All Years)

In this question we are investigating the straight line $11x+13y = 382$ (which we call equation *). A diagram of the line (not to scale) is shown in Figure 1.

(a) If we substitute $x = 0$ into the equation *, which point (A or B) do we find the co-ordinates of? A. Surprisingly poorly done. See below.

(b) Substitute $x = 0$ into the equation * to find the co-ordinates (in the form (a, b), where a and b are rational numbers) of the appropriate intercept. $(0, 382/13)$. 'Decimal' answers were accepted but '(0, 29)' was not unless the level of rounding was shown. Several students did not use parentheses.

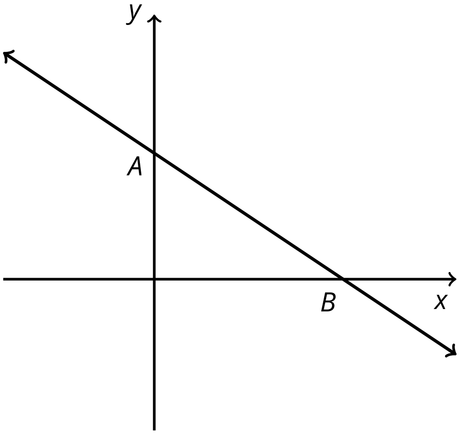


Figure 1 (not to scale)

A third variable t may be introduced to equation * to give the equivalent equation

$$11(2292 - 13t) + 13(-1910 + 11t) = 382.$$

In this version of the equation $2292 - 13t = x$ and $-1910 + 11t = y$.

(c) If $t = 0$ write down the values of x and y. $x = 2292$ and $y = -1910$. Well answered although many students had missed the whole question out.

(d) Show that if $t = 357$ then $y = 2017$ and find the corresponding value for x.

$y = -1910 + 11 \times 357$ (= 2017).
$x = 2292 - 13 \times 357 = -2349$.

Incorrect use of 'BEDMAS' led to many errors. For y working was needed for full credit.

(e) Solve for t the two inequalities

$$2292 - 13t > 0$$

and

$$-1910 + 11t > 0.$$

Give all integer values between the two solutions you have found. $2292 - 13t > 0$ gives $t < 176.30$ (1 d.p.), while $-1910 + 11t > 0$ gives $t > 173.6$ (1 d.p.). Thus t is one of 174, 175, or 176. As expected this caused problems, although some Year 9 students had no trouble.

(f) Use the solution to part (e) to find the co-ordinates of all points on the line * where x and y are both positive integers. $t = 174$ gives $(30, 4)$. $t = 175$ gives $(17, 15)$. $t = 176$ gives $(4, 26)$. Usually missed out, although many Year 11 students sped through the question. For those who could do (e) this was very easy.

Question 5 (All Years)

Moana designs a stylised flower inside a circle for a company logo. Each petal consists of a semicircle with radius a joined to an isosceles triangle with two leg sides of length b, a base side of length c, and a vertex angle of 30°. (Note that $c = 2a$.) A circle of radius r is drawn around the flower such that the curved edge of each petal just touches the circle's circumference. See Figure 2.

(a) What is the centre angle of four adjacent petals combined? 120°. Well answered.

(b) The logo in Figure 2 has 12 petals. If the vertex angle of each petal was 20° instead of 30°, how many petals would there be? Since $360/30 = 12$, it follows that $360/20 = 18$, and so there must be 18 petals. Well answered.

(c) Suppose each self-contained area of the logo is coloured in. Each self-contained area of the logo cannot have the same colour with an area adjacent to itself. How many different colours are needed? Briefly explain your answer. 3, since each petal is next to two other petals (which are not adjacent to each other), but each part of the circle not covered by the flower is adjacent to two petals, so must use a third colour. Explained well by about one third of candidates who attempted the question but for some otherwise excellent candidates this was suprisingly poorly done. There were many Year 11 students who answered all of Question 5 correctly except for this part.

For parts (d) and (e) assume $b = 3\,\text{cm}$.

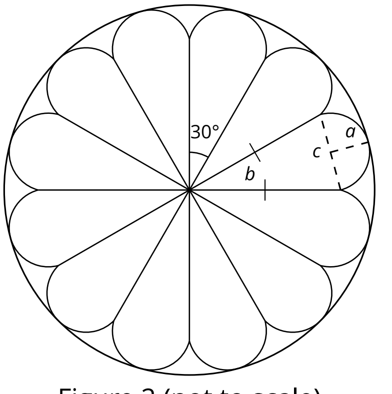


Figure 2 (not to scale)

(d) (i) Show that the length of c is 1.553 cm (to 3 decimal places). (Use this value of c from now on if you cannot find it yourself.) Half the vertex angle is 15°. Using the TOA part of SOHCAHTOA we get $c = 2O = 2 \times \sin 15° \times 3$, so $c = 1.553\,\text{cm}$ to 3 decimal places. You could also special triangles (ratio $1{:}\sqrt{3}{:}2$) and Pythagoras. Another viable alternative is to use the sine rule: $\frac{3}{\sin 75°} = \frac{c}{\sin 30°}$ etc. $6\cos 75°$ also reaches the answer. Not well answered by most, although many Year 11 students had no problems.

(ii) Hence find the area of an individual petal. Give your final answer to 3 decimal places but be as accurate as possible with your working. We can use Pythagoras to find the height (h) of our triangle : $h^2 + 0.7762 = 32$, so $h = 2.898$ to 3 decimal places. As such the area of each 'half' triangle is $\frac{1}{2} \times 0.776 \times 2.898 = 1.125\,\text{cm}^2$. (This number is exact.) So the area of two triangles (and thus the area of one isosceles triangle) is $2.25\,\text{cm}^2$.

Each semicircle can be found using $\frac{1}{2}\pi(1.553/2)^2 = 0.947\,\text{cm}^2$ to 3 decimal places. The area of each petal is therefore $2.25 + 0.947 = 3.197\,\text{cm}^2$ to 3 decimal places. Too difficult for most. The area of one of the semicircles (sometimes twice this) was the most frequently correct part seen here.

(e) Find the area of the outer circle not covered by the flower. Give your final answer to 3 decimal places but be as accurate as possible with your working. The area of the circle is $\pi(h + c/2)^2 = 13.5\pi\,\text{cm}^2$ or $42.412\,\text{cm}^2$ to 3 decimal places. (The 13.5 is exact.) The area of all twelve petals combined is $12 \times (h + c/2) = 12 \times (2.25 + 0.947) = 38.364\,\text{cm}^2$ to 3 decimal places. The area not covered by the flower is then $42.412 - 38.364 = 4.047\,\text{cm}^2$ to 3 decimal places. It is noticeable that trigonometry does not seem to be taught before Year 11 in many schools, although some younger students (including one Year 8 student) seem to have taught themselves.

Things Not To Do in Mathematics Exams - A Beginner's Guide

(a) If a question is really easy, don't use about six lines explaining it. For example in Question 1(a) all you had to do was add five numbers. Many students showed us how to do it in three different ways (and repeated it in 1(b)). A typical example was: '3 + 0 + 4 + 26 + 5 = 38. This is because 3 + 0 = 3, 3 + 4 = 7, 7 + 26 = 33, and 33 + 5 = 38. Another way of doing it is to go 3 + 0 = 3, 4 + 26 = 30, 5 = 5, so 3 + 30 + 5 = 38. So the answer is 38.' Many students who spent so long on the first question had not done very much when time was up.

(b) If you are told something don't write down that it must be incorrect. For example in 2(b) you were asked to show that 2009 is not prime. Despite the statement suggesting that 2009 *is not* prime, many hundreds of students divided 2009 by 2, 3, and 5 (only) and concluded that 2009 was prime after all. (In fact 2009/7 = 287.) Examiners do make mistakes, but in 99/100 cases they get it right.

(c) Don't answer questions out of order. A few students started with Question 5 and then found they did not have time for the 'easy' Question 2.

(d) Don't use words where Maths symbols are quicker. An example is 3(b)(ii) where all you needed to write was something like '2047/23 = 89'. Some students wrote long sentences. An example would be '2047 is not prime because if 2047 is divided by 23 the answer would be 89 which is a whole number.'

(e) Once you've found the answer don't try to find more. In 2(a)(iv) once you've found that 4035 divides by 5 to give a whole number don't try all the other possible numbers as well, spending half a page on it.

(f) Don't forget to number your answers. Unnumbered answers can make it hard for markers to work out if you are correct or not.

(g) Try not to give an impossible answer. In 2(c) you were asked to write down the only prime between 2006 and 2016. The students who wrote down 2003 had no chance of giving a right answer.

(h) If a question has two choices you should give one of them. In 4(a) you were asked to choose *A* or *B*. Many students wrote '*y*' and received no credit.