Department of Mathematics and Statistics

UNIVERSITY of OTAGO
Te Whare Wānanga o Otāgo
NEW ZEALAND

Junior Mathematics Competition 2016 Solutions and Comments

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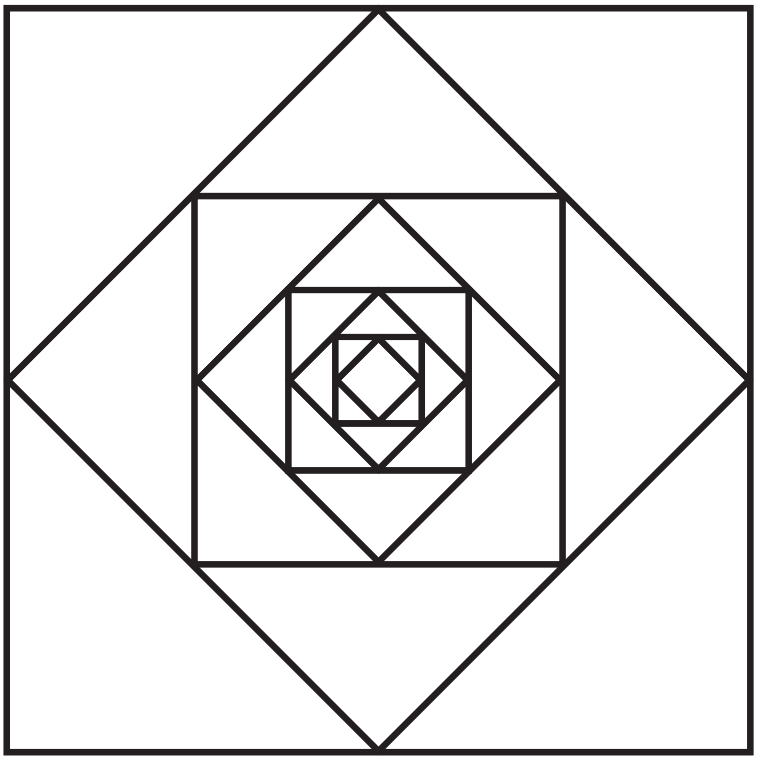

Year 9 (Form 3) Prize Winners

First Cameron Van Rynbach, Palmerston North Boys' High School

Second Grace Chang, St Kentigern College

Third Oliver Robinson, Hutt International Boys' School

Top 30 (in School Order):

Andrew Jia An Che, Auckland Grammar School
Luke Ying Feng Bao, Auckland Grammar School
Eddie Zhang, Auckland Grammar School
Kelvin Liu, Auckland Grammar School
Jordan Peters, Burnside High School
Matthew Gibb, Christ's College
Andrew Kwak, Christ's College
Victoria Sun, Epsom Girls' Grammar School
Chen Huan Liu, Epsom Girls' Grammar School
Ethan Qi, Glendowie College
Allan Taylor, Hillmorton High School
Chloe Yip, Howick College
Ishan Nath, John Paul College
Tobias Devereux, Kavanagh College
Johnathan Leung, Kings College
Vanessa Xiong, Kristin School
Kunli Zhang, Kristin School
Alphaeus Ang, Liston College
Haomo (Jasmine) Zhang, Macleans College
Terry Shen, Macleans College
Dillon Hong, Macleans College
Darsh Chaudhari, Macleans College
An Nguyen, Nelson College
Minju Kim, Rangitoto College
Ming Wang, St Cuthbert's College
Matthew Lang, Timaru Christian School
Tiana Mayo, Waikato Diocesan School for Girls

Year 10 (Form 4) Prize Winners

First William Han, Macleans College

Second Liam Wong, Auckland Grammar School

Third Honoka Osaki, Selwyn College

Top 30 (in School Order):

Serena Jou, ACG Strathallan College
Louie Wei, Auckland Grammar School
Jing Qu, Botany Downs Secondary College
Yash Shahri, Botany Downs Secondary College
Felix Backhouse, Burnside High School
Oscar Cunningham, Christchurch Boys' High School
Daniel Chong, Christchurch Boys' High School
Selena Fu, Diocesan School for Girls
Patrick Zhang, John McGlashan College
William Scharpf, King's High School
Marcus Ooi, Kings College
James Mead, Kings College
Alan Li, Lincoln High School
Joshua Zhu, Macleans College
Kevin Hou, Macleans College
Nicholas Yao, Macleans College
Michael Luo, Macleans College
Kevin Hu, Macleans College
Samuel Thompson, Otago Boys' High School
Songyan Teng, Pakuranga College
Robin Shen, Paraparaumu College
Hyeokjun Kwon, Roncalli College
Eric Li, St Kentigern College
Steven Cho, St Kentigern College
Connor Gallagher, St Peter's College (Epsom)
Jonah Smith, Taieri College
Alice Cao, Waikato Diocesan School for Girls

Year 11 (Form 5) Prize Winners

First Andrew Chen, St Kentigern College

Second Chuanye (Andrew) Chen, Auckland International College

Third Robin Wan, Burnside High School

Top 30 (in School Order):

Haozhe Wei, Auckland Grammar School
Max Gu, Auckland Grammar School
Hyeonmin Seo, Auckland Grammar School
Advait Pillarisetti, Auckland Grammar School
James Lobb, Auckland Grammar School
Ducksoo Shim, Auckland International College
Bowen Fan, Auckland International College
Naifan Chen, Auckland International College
Xutong (Tony) Wang, Auckland International College
Jeffry Chen Hung Chen, Auckland International College
Derek Long, Botany Downs Secondary College
Jordan See, Botany Downs Secondary College
Nick Peters, Burnside High School
Josh Hogan, Cambridge High School
John Voss, Hillcrest High School
Regan Chen, Kings College
Song Kang, Macleans College
Yang Fan Yun, Macleans College
Allen Hui, Macleans College
Owen Wang, Macleans College
Rohan Sadhu, Macleans College
Nick Wright, Napier Boys' High School
Russell Boey, St Andrew's College
Callum Lee, St Kentigern College
Yunfan Yu, St Kentigern College
Daniel Mar, St Kentigern College
Chanin Chungsuvanich, St Paul's Collegiate School
Eileen Kang, Westlake Girls' High School

2016 Answers

As always, only one method is shown. Often, several methods exist. We don't guarantee that the method shown is the best or fastest.

Question 1 (Years 9 AND BELOW ONLY)

Finn wants to buy a car to replace his old one. He drives from his home to the salesperson's yard.

(a) *On the way to the salesperson he travels at an average velocity of 25 km/hr. The entire trip takes 30 minutes. How far is it (in kilometres) from his home to the car salesperson's yard?*

25/2 = 12.5 km. Well answered, although some silly answers like 25 × 30 = 750 km appeared far too often. If he travels 25 km in one hour, surely he covers less distance in less than an hour? Notice how we answered the question in less than half a line. We marked numerous answers that took five lines or more. You'll never have time to answer everything if you take five lines (or more) to answer Question 1(a).

(b) *Finn cannot afford the $12 000 cash price for his new car. He is offered two alternatives for finance:*

(i) A deposit of $1500 and then nine equal monthly payments of $1300 each time.

(ii) A 10% deposit of the cash price and then nine equal monthly payments of $1320 each time.

Which is the better (cheaper) deal? Explain your reasoning.

(i) 1500 + 9 × 1300 = $13 200

(ii) 1200 + 9 × 1320 = $13 080

The second option is better. Well answered in the main. A few students had trouble with the order of operation, doing the addition before the multiplication.

(c) *After he has bought the new car, he goes for a drive on the open road. He travels at an average velocity of 80 km/hr. What is this in m/sec (metres per second)?*

80 km/hr = 80 × 1 000 / 3600 m/s

= 22.22 . . . m/s

Answered well by about half the candidates. However, this was where some ridiculous answers first appeared, like 288 000 000 m/s (almost the speed of light). One student wrote a ridiculous answer, but then also wrote 'far too high – haven't got time' and received bonus credit for showing common-sense.

(d) *When you are driving on the open road, it is recommended that you stop and rest every 100 km and <u>also</u> every two hours. Finn goes for a trip the following weekend. Travelling at a velocity of 80 km/hr on a 420 km journey, how many times should he stop and rest before the end of his journey if he follows all of the recommendations? Show your reasoning. (Do not count the pause times in your calculations.)*

Travelling at 80 km/hr, it takes 1.25 hours to cover 100 km.

100 km: stop after 1.25 hours	2 hours: stop after 160 km
200 km: stop after 2.5 hours	300 km: stop after 3.75 hours
4 hours: stop after 320 km	400 km: stop after 5 hours

You need six stops. Reasonably well answered. Some students said things like 5.5 rests. We don't know how you can have half a rest.

(e) *On his way home, he passes another car. Travelling at 98 km/h he pulls out to pass. It takes two and a half (2.5) minutes to completely pass the other car. How far has he travelled in that time? Give your answer to the nearest tenth of a kilometre.*

98 × 2.5/60 = 4.08333 . . .

= 4.1 km (nearest 10^{th} of a km).

Fairly well answered. One student said it was an unreasonable answer, but we've worked it out, and if the other vehicle is travelling at nearly Finn's speed (say 96 km/hr), it could take approximately 4 km to complete the passing manoeuvre. There are possibilities for a project here.

Question 2 (All students)

2016 is a leap year, which has 366 days (as opposed to the normal 365 days). A leap year is defined as any year divisible by 4, unless it is divisible by 100 and not 400.
For example, 2000 (a multiple of 400), 2020, and 1840 are leap years, while 1969 (odd), 1766 (not a multiple of 4), and 1900 (a multiple of 100 but not 400) are not leap years.

(a) *For each of the following statements, answer true or false:*

(i) *1956 is a leap year.* True

(ii) *1800 is a leap year.* False

(iii) *1986 is a leap year.* False

Generally well answered, although some people, who didn't read the information given above, thought that 1800 was a leap year. However, 1800 is divisible by 100 but not 400.

(b) *In the period 1816 to 2016 (inclusive), how many leap years were there?*

Every multiple of 4 in this period except 1900 is a leap year. There are 25 multiples of 4 per 100 years, and we have a period of 201 years. Thus 50 leap years (25 × 2 - 1 + 1).

Not well answered. Most students subtracted 1816 from 2016 (sometimes not very successfully) and got 200, which they divided by 4 (also not always correctly) to get 50. Right answer; wrong working. They failed to account for the fact that both end values were leap years (an 'out-by-one' error) and then they also had to account for 1900.

(c) *What was the last leap year prior to 2016 (since 1600) which was also a multiple of 30?* 1980

Mixed results. 2010 was a common answer, but it was <u>not</u> a leap year.

(d) *Calculate the average number of days any year contains under the present leap year system.*
Give your answers to four decimal places and show necessary working.

To get the correct answer, you needed to consider a full 'cycle' of 400 years. From 1601 to 2000 there were 97 leap years and 303 'normal' years. (1700, 1800, and 1900 were 'normal'.)

(303 × 365 + 97 × 366) / 400 = 365.2425

Not well answered. A good predictor of success overall. Almost everyone who answered this question correctly gained Merit or better, while those who didn't read the question well, used their own 'knowledge' and gave the incorrect 'answer' 365.25 (or 365.2500), did not always do well overall.

(e) *For how many days did the Māori Queen Dame Te Atairangikaahu (23 July 1931 – 15 August 2006) live altogether?*

- From 1931 to 2005, there were 75 years in which there were 19 leap years and 56 'normal' years (you can list the leap years if you like):
  19 × 366 + 56 × 365 = 27 394
- In addition, she lived from 23 July to 15 August, a period of 24 days inclusive.
- 27 394 + 24 = 27 418 days (inclusive).

The single question in the whole competition giving the highest amount of credit. Mixed results. We gave partial credit to anyone scoring plus or minus five days. We had no sympathy for the student who wrote 'don't know how many days in each month', and then missed the whole question out. If instead he had guessed 30 (July actually has 31), he might have reached 27 417 and received good credit.

A lot of students appeared to spend a lot of time on this part of the question, to the extent that they ran out of time to do the rest of the competition. Sometimes identifying questions that are hard and doing other questions instead is a better approach to doing well, although doing such is a skill in itself!

Question 3 (All Students)

2016 marks the thirtieth anniversary of the Junior Mathematics Competition which began in 1986. In this question, we revisit some old questions (with changes).

PART A

The number 10 can be made by using each of the numbers 1, 9, 8, 6 exactly once and any of the standard operations +, − , ×, ÷, √, which may be repeated as necessary or not used at all, and any number of brackets. For example, 10 = 9 + 8 – 6 – 1.

(a) *Use the same method to construct each of the numbers*

(i) 20 = (9 + 1) × (8 – 6)

(ii) 30 = (√9 × 8 + 6) × 1

Mixed results. Some used some of the numbers more than once; some introduced 'extra' functions (like squaring). There are additional answers, which are left as an exercise.

(b) *We can construct the number 10 using the numbers 2, 0, 1, 6 in the same way, since 10 = (6 - 1) × 2 + 0. Is it possible to construct 20 in the same manner? If it is, show your construction. If not, briefly explain why it is impossible.*

It is not possible. The largest number we can construct is 18 = (2 + 1) × 6 + 0, which is smaller than 20.

Again mixed results. Students had to mention '18' for full credit. Some students could answer this part correctly, but somehow could not do part (a) correctly.

PART B

The number 1996 has six factors: 1, 2, 4, 499, 998, and 1996.

(c) *List the factors that 1996 shares with the numbers*

(i) *1986:* 1 and 2

(ii) *2006:* 1 and 2

(iii) *2016:* 1, 2, and 4

Most students who attempted the question answered it correctly.

(d) *How many factors does the number 2016 have? (Hint: the number 1996 can be written as $2^2 \times 499$, where 2 and 499 are primes. Write 2016 in the same way.)*

There are 36 factors in total. $2016 = 2^5 \times 3^2 \times 7$. Thus 6 possibilities for powers of 2, 3 possibilities for powers of 3, and 2 possibilities for powers of 7. Hence 6 × 3 × 2 = 36.

Mixed results. If students listed all the factors and reached 36, they received full credit.

PART C

The numbers 1986 and 2006 share the property that when divided by 5 the remainder is 1, and when divided by 4 the remainder is 2.

(e) *Explain why there are no numbers between 1986 and 2006 with the above property.*

The numbers between 1986 and 2006 which have remainder 1 on division by 5 are 1991, 1996, and 2001. The first and last are odd, so cannot have remainder 2 on division by 4. 1996 is a multiple of 4, and the result follows.

A more complicated approach uses the Chinese Remainder Theorem (which a couple of students explicitly named). This is a classic example of the simpler approach being better however, as more complicated mathematics wastes time and is more error prone.

(f) *Find non-negative integers a, b, p, and q (with a and b greater than 1 and a not equal to b) such that for any member y of the set {1986, 1996, 2006, 2016} we have remainder p on division of y by a, and remainder q on division of y by b.*

(The numbers a, b, p, and q are the same for any value of y.)

We can take a and b uniquely from the set {2, 5, 10}. So there are only 3 possibilities (ignoring order):

$a = 5$, $b = 2$, $p = 1$, $q = 0$.

$a = 5$, $b = 10$, $p = 1$, $q = 6$.

$a = 2$, $b = 10$, $p = 0$, $q = 6$.

Not commonly answered. A lot of candidates struggled to understand the question, but it was intended to be hard.

Question 4 (All Students)

(a) *If a pie weighs 1400 grams, how much will a slice of pie weigh if it is shared equally by weight between 4 people?*

1400 / 4 = 350 grams. Well answered.

(b) *Suppose one person takes a 500 gram slice from a 1400 gram pie. If the remaining pie is shared equally by weight between 3 people, how much will each slice of the pie weigh?*

1400 - 500 = 900

900 / 3 = 300 grams each. Well answered.

(c) *Suppose three fifths of a pie sliced into four equal quarters is covered evenly with walnuts, as shown in Figure 1 (the portion in white has no walnuts, the darker portion does). If a slice of pie is partially covered with walnuts, what percentage of that slice is covered in walnuts?*

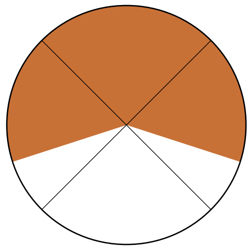

Figure 1

3/5 - 1/4 = 12/20 - 5/20

= 7/20 amount of pie left after the slice with walnuts is removed.

This amount is shared between two slices, but each slice is a quarter of the pie. So (7/20 ÷ 2) × 4 = 7/10 per slice. Hence 70% of each such slice is covered in walnuts.

Mixed results, but often seen. The ability to subtract fractions was important, although many converted the fractions into decimals, then subtracted.

(d) *Suppose a 1400 gram pie is to be shared between 4 people, with three people taking an equivalent slice of pie by weight, and the fourth taking a slice of pie half the weight of the other slices. How much does each slice weigh?*

We have 1/a + 1/a + 1/a + 1/2a = 1. Thus 2 + 2 + 2 + 1 = 2a. So 2a = 7, so a = 7/2. Thus the three big slices are each two sevenths of the pie, leaving one seventh of the pie, half as heavy as the first three. Thus the bigger three slices weigh 400 grams each, while the smaller slice weighs 200 grams.

The answer was often seen, for partial credit. But for full credit we needed, at the very least, to see evidence of '7' (e.g. '$7x = 1400$').

(e) *A circular cake dish with a total diameter of 25 cm has a circular hole 1/8 of the total diameter cut into the centre, into which no ingredients can fall. What is the total volume of a cake if it completely fills the dish to a height of 10 cm?*

$V = \pi r^2 h$

$= \pi(12.5^2 - (25/16)^2)10$

$= 4832 \text{ cm}^3$ (approximately)

Often well answered. But far too many from Year 11 had no idea. This was as close to NCEA as the competition gets. A = π d h or some similar incorrect formula may be expected from Year 9 (and below) students (often even π was missing), but how to handle this type of problem should be 'standard' by Year 11. Poor order of operation (e.g. incorrectly multiplying by π before squaring) was often seen. Some Year 8 students however succeeded in the question!

Comment: By this stage, many students had run out of time. We don't think that this year the competition was very hard (many students might disagree), but we admit it was long. 'Perfect' answers to Question 5 were rarely seen, although excellent answers to all parts except the last part weren't uncommon. One Year 9 student gave correct answers to all parts of Question 5, but he s*howed no working*! He received a Merit, but could have been first if he had only *shown working*!

Question 5 (All Students)

(a) (i) *Inside a square of side length 10cm is drawn another square, such that the vertices of the sides of the smaller square touch the midpoints of the sides of the larger square (see Figure 2). Find the area of the smaller square.*

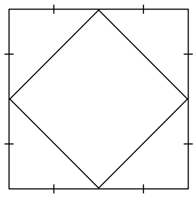

Figure 2

Area = ½ (product of diagonals)

= 50 cm^2.

The formula for the area of a rhombus is not well known, or if known, was not applied. Many students took about eight lines to answer this, usually finding the area of the corner triangles, then subtracting from 100. Rounding errors were common. Quite a few students found a side length of √50, which they rounded to 7. They then told us the area was 49 cm^2.

(ii) *An even smaller square is drawn inside the second square on the same principles – the vertices of the third square touch the midpoints of the sides of the second square (see Figure 3). Find the area of the third square.*

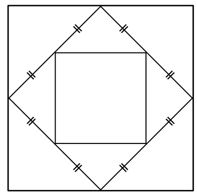

Figure 3

25 cm^2. If the previous answer was found, then usually (but not always) this answer was found too.

(iii) *This process is repeated as long as possible. How many times altogether (i.e. starting at the beginning) do you need to carry out the process until you reach a square with area less than 1 cm^2?*

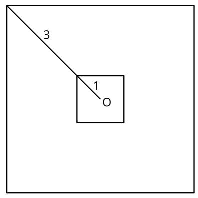

Figure 4

The table shows the pattern:

Iterations	1	2	3	4	5	6	7
Area	50	25	12.5	6.25	3.13	1.56	0.78

It takes 7 times for the area to be less than 1 cm^2 .

An 'out-by-one' error was common, counting the first (100 cm^2) square and reaching the incorrect answer of '8'.

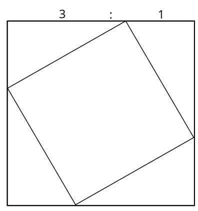

Figure 5

(b) *A square is drawn inside another square of side length 10cm so that the radius of the larger square is divided in the ratio 3:1 (see Figure 4). Find the area of the smaller square.*

Enlargement scale factor 1/4. Area = 100/16 cm^2 or equivalent (6.25).

'Difficult', but often seen from those who got this far.

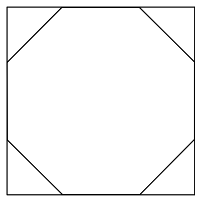

Figure 6

(c) *A square is drawn inside another square of side length 10cm so that the vertices of the smaller square divide the sides of the larger square in the ratio of 3:1 (see Figure 5). Find the area of the smaller square.*

Area of triangles = 4 × (½ × 7.5 × 2.5)

= 37.5

So the area of the smaller square = 62.5 cm^2.

Again, often seen from those who reached this far.

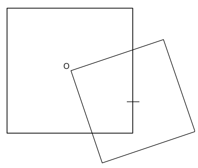

Figure 7

(d) *An octagon is drawn inside a square of side length 10cm. Four sides of the octagon meet the sides of the square and divide the sides of the square into thirds (see Figure 6). Find the area of the octagon.*

Area of triangles = 4 × ½ × (10/3)^2

= 200/9

So the area of the octagon = 100 – 200/9

= 700/9 or 77.8 (1 d.p.) cm^2.

Not actually uncommon, especially from Year 11 candidates.

(e) *A square is drawn alongside another square of side length 10cm such that a vertex of the smaller square lies at the centre O of the larger square. The centre of the smaller square lies on the right hand edge of the first square, one quarter of the way up the edge (see Figure 7). What is the area of the overlap between the squares?*

Extend the sides out as shown in Figure 8. By rotational symmetry the overlap is 25 cm^2. (Each of the four quadrilaterals forming the larger square has the same area.)

Hardly seen (with working), but by this stage most students had run out of time.

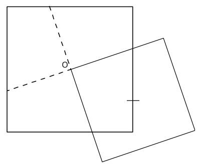

Figure 8

Common-sense and Estimates

When the competition manager was at Secondary School in the 1960s, he was always being told that first he should estimate his answer, so that his final answer might make sense. Let's apply that to the question about the number of days the Māori Queen lived. If you remember, it said that she lived from 23 July 1931 – 15 August 2006. If you take 2006 and subtract 1931, you get 75 (years). Multiply this by the nearest easy number to 365, i.e. 400. Now 75 × 400 = 30 000 (many of you should be able to do this in your head), so that you know before you start the question that she managed to live for less than 30 000 days. If you then get a 'stupid' answer (the silliest we got was 3 370 003 841 days, which is over nine million years – we also had answers over 6800 and 20 200 years), you know you have to redo the question if you've got time.

Some estimates are done in the head. How many rests are needed on a 420 km car trip? It says you should rest every 100 km, so an estimate of four rests is a good starting point. If you then get 8404 rests (somebody did), you then know that you're well out.

Similiarly, in Question 5 (a)(i), how big is a square inside a larger 100 cm^2 square? It looks to be about half the big square, so 50 is a quick visual estimate (this time the estimate turned out to be accurate). So if you get 40 000 cm^2, or even 500 cm^2 (we received both – the 500 cm^2 answer was 'common'), you know that you have to look at the question again.

Reading the Question

Obviously, this is important. An example occurred in Question 2. A 'leap year' (with examples) was defined at the top of the question. Many students glanced at it, thought 'I know all about leap years', and carried on carelessly to receive a poor mark in the question because, no, they didn't know the correct definition of a leap year after all. Moral: **carefully read the question**. Usually, it contains vital information which you will need.

Simpler is Often (But Not Always) Better

In some cases, it is better to use brute-force than to use elegant mathematics. For example, in Question 3 (e) it is generally better to find the three numbers between 1986 and 2006 that have remainder 1 on division by 5, and show that none have remainder 2 on division by 4. Using more "generalised" techniques means you have to choose your words carefully. Many answered this question by pointing out that 5 × 4 = 20, and 2006 - 1986 is also 20. This approach only works since the Least Common Multiple of 5 and 4 is also 20. If we had chosen 6 and 4 instead (with appropriate years to match), this method quickly breaks down. (Why?)

A good rule of thumb is that if your approach involves listing more than about 10 things, it isn't a good approach!