Department of Mathematics and Statistics

UNIVERSITY of OTAGO
Te Whare Wānanga o Otāgo
NEW ZEALAND

Junior Mathematics Competition 2021

Questions for Part 2

Instructions to Candidates

You have a maximum of **one** hour to answer **six** questions out of **eight**. The set of questions you answer is determined by your year level:

Question 1: 10 marks. Year 9 and below only.
Question 2: 10 marks. Year 10 and below only.
Question 3 to Question 6: 20 marks each. All students.
Question 7: 10 marks. Years 10 and 11 only.
Question 8: 10 marks. Year 11 only.

If you answer an incorrect question for your year level it will not be marked.

These questions are designed to test your ability to analyse a problem and express a solution clearly and accurately.

Please read the following Instructions carefully before you begin.

1. Do as much as you can. You are not expected to complete the entire paper. In the past full answers to three full (20 mark) questions have represented an excellent effort.
2. You must explain your reasoning as clearly as possible with a careful statement of the main points in the argument or the main steps in the calculation. Generally even a correct answer without any explanation will not receive more than half credit. Likewise clear and complete solutions to three full problems will generally gain more credit than sketchy work on four.
3. Credit will be given for partial solutions and evidence of a serious attempt to tackle a problem.
4. Textbooks are NOT allowed. Calculators may be used and students who do not have one may be disadvantaged. Otherwise normal examination conditions apply.
5. We recommend black or blue pens. Dark pencil is acceptable if you have nothing else. Do NOT use red or green pens, or light pencil that we cannot read.
6. We will penalise inappropriate rounding and incorrect or absent units.

DEFINITION

A *prime number* has exactly two factors, 1 and itself. By this definition, 1 is not a prime number.

The first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29.

DO NOT TURN OVER UNTIL TOLD TO DO SO.

Question 1: 10 marks (Years 9 and below only)

You do not need to show working in this question. Note that a perfect square is of the form $n \times n$ and a cube is of the form $n \times n \times n$, where n is an integer in both cases.

If $a = 1$, $b = 2$, $c = 3$, and so on up to $z = 26$, use the following clues to determine a mystery word. For each clue write down the associated number. At the end write down your word. (All letters map to a number between 1 and 26 inclusive.)

(a) My first letter is equal to the positive square root of 49.

(b) My second and fifth letters are both equal to the smallest sum of two different primes.

(c) My third letter is equal to three times my second letter.

(d) My fourth letter is equal to the sixth prime number.

(e) My sixth letter is greater than my seventh number, smaller than 26, and is equal to a prime times a perfect square.

(f) My seventh letter is equal to twice the square of my ninth letter.

(g) My eighth letter is equal to the first odd number one greater than a non-zero positive cube number.

(h) My ninth letter is equal to the first odd prime number.

Question 2: 10 marks (Years 10 and below only)

Jane is building her dream home. She has a section of 500 square metres on which to build. She considers two plans for her house:

> *(1) A three storey house with a ground floor area of 380 square metres.*
>
> *(2) A two storey house with a ground floor area that covers 87.5% of her section.*

(a) What percentage of Jane's section will plan (1) take up?

(b) What will the ground floor area of Jane's house be in square metres if she chooses plan (2)? Your answer should be given to two decimal places.

(c) Suppose Jane chooses plan (1) for her house. If the first floor is 75% of the area of the ground floor and the second floor is 75% of the area of the first floor, what is the area of the second floor in square metres? Your answer should be given to two decimal places.

(d) If the first floor of the house in plan (2) has the same floor area as the ground floor, will the total floor area (the sum of the floor area of each floor) be bigger in plan (1) or plan (2)? Write the total floor area for both plans as part of your answer.

Question 3: 20 marks (All Years)

At Kakanui University a regular undergraduate degree takes three academic years. Each paper in the degree is worth a different number of points (either 5 points, 9 points, or 12 points), with the final degree being worth a total of 180 points.

(a) If each academic year is made up of two *semesters*, what is the average number of points needed per semester to finish the undergraduate degree in three years?

(b) If each academic year is instead made up of three *trimesters*, what is the average number of points needed per trimester to finish the undergraduate degree in three years?

For the rest of the question we assume each academic year is made up of two semesters. Each semester students must do a minimum of 24 points worth of papers, and a maximum of 32 points worth of papers, and each student must do papers every semester in the three academic years.

(c) If a student does exactly one semester of precisely 24 points in a three year undergraduate degree, what is the average number of points that student must do in the rest of the semesters needed to finish the degree?

(d) What is the maximum number of semesters with precisely 32 points in them that a student may have in a three year undergraduate degree?

(e) What is the maximum number of 9 point papers a student may take in a three year undergraduate degree?

Question 4: 20 marks (All Years)

In this question the *resolution* of a rectangular screen or photo is written as $w \times h$, where w is the width in pixels and h is the height in pixels. For example a screen of resolution 800×600 is 800 pixels wide and 600 pixels high, and has a total of 480000 pixels.

Achara is designing a photo editing website. Some photos have to be scaled to a certain width and height to fit on the various screen sizes Achara anticipates people will use. (Photos that are smaller than the resolution of a given screen do not need to be scaled.) The aspect ratio of each photo will be preserved in all cases.

For a photo of resolution $a \times b$ and (smaller) screen of resolution $c \times d$, the *scale factor s* is the largest number between 0 and 1 such that $as \leq c$ and $bs \leq d$. For example if a photo of resolution 3840×2160 is to be scaled to fit a screen of resolution 1920×1080, the *scale factor s* will be 0.5.

(a) How many pixels are there in a screen of resolution 1280×1024?

(b) Consider a photo of resolution 1920×1080.

   (i) What scale factor s is needed to scale our photo to fit a screen of resolution 1280×720? Round your answer to three decimal places.

   (ii) What scale factor s is needed to scale our photo to fit a screen of resolution 1024×768? Round your answer to three decimal places.

Achara decides that putting a small border of 10 pixels around each photo will look better on a given screen.

(c) If a photo of resolution 1920×1080 is scaled to fit a screen of resolution 1024×768 and a border of 10 pixels is required, to three decimal places what is the scale factor needed?

One of the ways people can edit photos is to crop them — that is, they make a smaller photo by selecting a particular rectangular part of the larger photo.

(d) If a photo of resolution 1920×1080 is to be cropped to a resolution of 1024×768, how many pixels will be removed in total?

Question 5: 20 marks (All Years)

Let $\sigma(n)$ be the sum of the divisors of a natural number n (the divisors include 1 and n). For example, if $n = 14$ then $\sigma(14) = 1 + 2 + 7 + 14 = 24$.

(a) For the following n find $\sigma(n)$:

   (i) $n = 6$.

   (ii) $n = 13$.

   (iii) $n = 30$.

We call a natural number d *deficient* when $\sigma(d) < 2d$, while a *perfect* natural number r has $\sigma(r) = 2r$, and an *abundant* natural number a has $\sigma(a) > 2a$.

(b) Find the smallest deficient number greater than 30.

(c) Find the largest perfect number smaller than 30.

(d) Find an abundant number between 41 and 50 inclusive.

(turn over)

Question 5 continued

All natural numbers can be classified as either deficient, perfect, or abundant.

(e) Briefly explain why every prime number is deficient.

(f) For each number between 21 and 30 inclusive, specify if it is deficient, perfect, or abundant. (Include 21 and 30 in your list; you do not need to show working in this part.)

Question 6: 20 marks (All Years)

For the purposes of this question, assume that a leap year occurs every four years, and such years have 366 days in them, with a day added in February. (Years that are not leap years have 365 days in them.) 2020 is a leap year. Also note that $7 \times 52 = 364$.

Rawiri (whose birthday is in June) goes to the gym every Tuesday and Thursday. Assume that he lives for a very long time.

(a) In 2019 Rawiri went to the gym on a Tuesday on his birthday in June. In what year could he first go again to the gym on a Tuesday which is his birthday?

(b) In 2020 Rawiri went to the gym on a Thursday on his birthday in June. In what year could he first go again to the gym on a Thursday which is his birthday?

(c) In what two consecutive years (after 2020) could Rawiri first go again to the gym on a Tuesday birthday and then on a Thursday birthday the year after?

Question 7: 10 marks (Years 10 and 11 only)

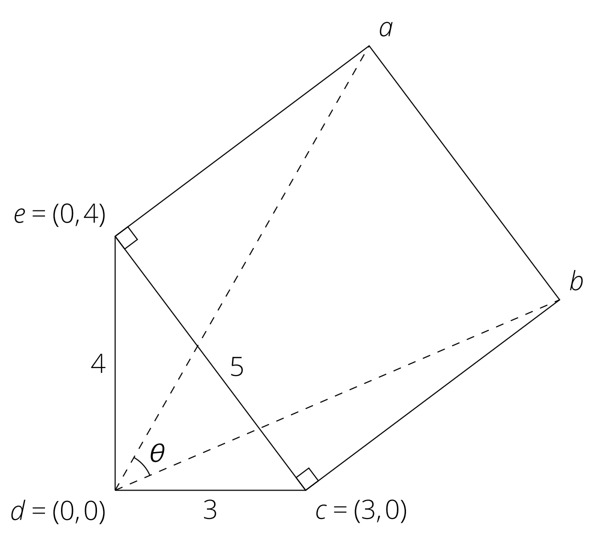
Suppose for some θ where $0° < \theta < 90°$ there exists a real number m such that $\tan\theta = \dfrac{(1+m)}{(1-m)}$.

(a) Find $\cos\theta$ in terms of m.

(b) Find the allowable values of m.

Question 8: 10 marks (Year 11 only)

It was a nice day, so Scott decided to take his class of gifted students to the gardens for a geometry lesson. He sat his class down at point d, from which everyone could see four fine trees located at a, b, c, and e, which defined the corners of a grass square with side length 5 metres.



(a) If Scott asks his class to find as co-ordinates the points a and b, what do they write down?

(b) From the group's vantage point of d, what is the angle θ between a and b? Round your answer to three decimal places.

(c) Find the area of the triangle defined by a, b, and d.

(END OF COMPETITION)