Department of Mathematics and Statistics

UNIVERSITY of OTAGO
Te Whare Wānanga o Otāgo
NEW ZEALAND

Junior Mathematics Competition 2020

Questions

Instructions to Candidates

You have **one** hour to answer **six** questions out of **eight**. The set of questions you answer is determined by your year level:

Question 1: 10 marks. Year 9 and below only.
Question 2: 10 marks. Year 10 and below only.
Question 3 to Question 6: 20 marks each. All students.
Question 7: 10 marks. Years 10 and 11 only.
Question 8: 10 marks. Year 11 only.

If you answer an incorrect question for your year level it will not be marked.

These questions are designed to test your ability to analyse a problem and express a solution clearly and accurately.

Please read the following Instructions carefully before you begin.

1. Do as much as you can. You are not expected to complete the entire paper. In the past full answers to three full (20 mark) questions have represented an excellent effort.
2. You must explain your reasoning as clearly as possible with a careful statement of the main points in the argument or the main steps in the calculation. Generally even a correct answer without any explanation will not receive more than half credit. Likewise clear and complete solutions to three full problems will generally gain more credit than sketchy work on four.
3. Credit will be given for partial solutions and evidence of a serious attempt to tackle a problem.
4. Textbooks are NOT allowed. Calculators may be used and students who do not have one may be disadvantaged. Otherwise normal examination conditions apply.
5. We recommend black or blue pens. Dark pencil is acceptable if you have nothing else. Do NOT use red or green pens, or light pencil that we cannot read.
6. We will penalise inappropriate rounding and incorrect or absent units.

DEFINITION

A *prime number* has exactly two factors, 1 and itself. By this definition, 1 is not a prime number.

The first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29.

DO NOT TURN OVER UNTIL TOLD TO DO SO.

Question 1: 10 marks (Years 9 and below only)

Mr Quick, a dangerous driver, was driving 660 km from Auckland to Wellington in just 6 hours. This required 62.7 litres of petrol, and when he arrived his tank was empty.

(a) What was his average speed?

(b) How much petrol on average was needed per 100 km?

(c) With an average speed of 90 km/h, the car would only have used 8.5 litres of petrol per 100 km. What additional distance could Mr Quick drive with the saved petrol at this speed? (Round your answer to the nearest integer.)

Question 2: 10 marks (Years 10 and below only)

Methane is a greenhouse gas that is about 50 times more destructive to the environment than carbon dioxide. In New Zealand, methane from cattle digestion accounts for almost one third of the country's greenhouse gas emissions. An average cow releases about 100 kg of methane each year. Recently, a New Zealand science organisation started experimenting with a food supplement from a native seaweed. First tests show that just a small quantity of this supplement could reduce stock methane emissions by 80%!

(a) New Zealand cows release about 500 million kg methane each year. How many cows do we approximately have in New Zealand?

(b) If every cow in New Zealand received the food supplement, how many kilograms of methane less would be burped out each year?

(c) In order to supply the growing world population with dairy and beef, even more cows will be needed in the future. However, with the food supplement New Zealand could have more cows and still reduce the release of methane into the environment. How many cows could we have if all of them received the food supplement and we wanted to reduce the methane emissions to 30% of the current value?

Question 3: 20 marks (All Years)

In this entire question you do not need to show working.

Every number is interesting in some way. In this question we will look at the interesting number 41. Firstly, it is a prime number, meaning that it has only two factors, 1 and 41.

(a) 41 is the sum of two *perfect squares*. Find them. (A perfect square is the square of an integer, such as $8^2 = 64$.)

(b) 41 is a *twin prime*, separated from another prime number by exactly 2. Find the prime number which is a twin to 41.

(c) 41 can be written as the sum of three different prime numbers in a variety of ways. Find one such way.

(d) 41 is the sum of exactly six different prime numbers. Find them.

(e) 41 can be found using the formula $[(2n-1)^2+1]/2$, where n is a natural number. Find the value of n that gives 41 using this formula.

(f) A prime p is a *Sophie Germain prime* if the number $2p+1$ is also prime. Thus if 41 is a Sophie Germain prime then $2 \times 41 + 1 = 83$ must also be prime. Is 83 prime? If it is, then say 'yes'. If it is not say 'no' and write down the factors of 83.

(g) For many years it was thought that some Mathematical expressions could be used to find prime numbers. For example, the expression $n^2 - n + 41$ (where n is a natural number) gives many primes. So $5^2 - 5 + 41 = 61$, and 61 is prime.

  (i) What value is given by the expression $n^2 - n + 41$ if $n = 35$?

  (ii) If $n = 41$, then $41^2 - 41 + 41 = 1681$. Is 1681 prime? If it is then say 'yes'. If it is not say 'no' and write down one factor of 1681 (that is not 1 or 1681).

Question 4: 20 marks (All Years)

The numbers

$$1,\ 3,\ 6,\ 10,\ 15,\ 21,\ \ldots$$

are called *triangular numbers*.

(a) State the values of the 8^{th} and 9^{th} triangular numbers.

(b) The triangular numbers can be listed with the natural numbers as a set of ordered pairs (x,y),

$$\{(1,1),\ (2,3),\ (3,6),\ (4,10),\ (5,15),\ (6,21),\ \ldots\}.$$

The formula for the x^{th} triangular number is a quadratic. In other words $y = ax^2 + bx + c$, where a, b and c are real numbers.

It can be shown that $c = 0$.

(i) If $x = 1$, write an equation for a and b.

(ii) If $x = 2$, write an equation for a and b.

(iii) State the values of a and b.

(iv) State the value of the 50th triangular number.

(c) Show that for any triangular number y, the number $32y + 4$ is always square.

Question 5: 20 marks (All Years)

In a certain school sports team, all students are either 13 or 14 years old (and there is at least one student in each age group). The sum of the ages of all students is 325.

(a) Denote the number of 13-year old students by x and the number of 14-year old students by y. Write down an equation for x and y from the information given.

(b) What are the prime factors of 325?

(c) Using (a) and (b), find out how many 13-year old and how many 14-year old students there are in the team, i.e. find the values of x and y.

Question 6: 20 marks (All Years)

Archie brings home a box of cherries for his three grandchildren Joe, Cathy, and Sally, which should be shared fairly between them.

Joe, who was alone at home, is the first to take his portion: he takes one third of the cherries from the box.

Then Cathy comes home. She does not know that Joe already took his cherries, so she takes one third of the remaining cherries.

Finally, Sally takes another third of the remaining cherries.

In the end, 16 cherries are left in the box.

How many cherries did each child take?

(turn over)

Question 7: 10 marks (Years 10 and 11 only)

A *Pythagorean triple* (a, b, c) consists of three positive integers a, b, c such that $a^2 + b^2 = c^2$.

(a) Find three different examples of Pythagorean triples with $a < b$ and confirm that they satisfy $a^2 + b^2 = c^2$.

(b) Why is there no Pythagorean triple with $a = b$?

Question 8: 10 marks (Year 11 only)

John locked his bike with a 4-digit combination lock several months ago, and he cannot recall the correct combination. Each digit is one of the numbers 0, 1, 2, ..., 9.

John only knows that each of the numbers 1, 4, and 6 appears exactly once, but he can't remember the position of those numbers, and he does not know what the fourth number was.

(a) What is the maximum number of combinations he would need to try to open the lock?

(b) If John knew that the first digit was the 4, how many combinations would he then need to try at most?

(END OF COMPETITION)