Department of Mathematics and Statistics

UNIVERSITY of OTAGO
Te Whare Wānanga o Otāgo
NEW ZEALAND

Junior Mathematics Competition 2019

Questions

Instructions to Candidates

You have **one** hour to answer **six** questions out of **eight**. The set of questions you answer is determined by your year level:

Question 1: 10 marks. Year 9 and below only.
Question 2: 10 marks. Year 10 and below only.
Question 3 to Question 6: 20 marks each. All students.
Question 7: 10 marks. Years 10 and 11 only.
Question 8: 10 marks. Year 11 only.

If you answer an incorrect question for your year level it will not be marked.

These questions are designed to test your ability to analyse a problem and express a solution clearly and accurately.

Please read the following Instructions carefully before you begin.

1. Do as much as you can. You are not expected to complete the entire paper. In the past full answers to three full (20 mark) questions have represented an excellent effort.
2. You must explain your reasoning as clearly as possible with a careful statement of the main points in the argument or the main steps in the calculation. Generally even a correct answer without any explanation will not receive more than half credit. Likewise clear and complete solutions to three full problems will generally gain more credit than sketchy work on four.
3. Credit will be given for partial solutions and evidence of a serious attempt to tackle a problem.
4. Textbooks are NOT allowed. Calculators may be used and students who do not have one may be disad- vantaged. Otherwise normal examination conditions apply.
5. We recommend black or blue pens. Dark pencil is acceptable if you have nothing else. Do NOT use red or green pens, or light pencil that we cannot read.
6. We will penalise inappropriate rounding and incorrect or absent units.

DEFINITION

A prime number has exactly two factors, 1 and itself. By this definition 1 is not a prime number. The first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29.

QUADRATIC FORMULA

If $ax^2 + bx + c = 0$ where $a \neq 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

DO NOT TURN OVER UNTIL TOLD TO DO SO.

Question 1: 10 marks (Years 9 and below only)

(a) Write A or B for the correct answer. You do not need to show working.

(i) Tea costs \$NZ3.29 a packet. Baked beans cost \$NZ1.59 a can. Which is cheaper?

(A) 1 packet of tea and 4 cans of baked beans, *or*

(B) 2 packets of tea and 3 cans of baked beans.

(ii) Which is the cheaper deal per 100g?

(A) 400g of cheese at \$NZ4.99, *or*

(B) 1 kg of the same cheese at \$NZ11.49.

(iii) On a particular day the exchange rate for the United States dollar is given by: \$US1 = \$NZ0.696. What is \$NZ500 worth in \$US to the nearest dollar?

(A) \$US348, *or*

(B) \$US718.

(b) If 3kg of flour costs \$NZ9.60 how much does it cost to buy 7kg in \$NZ?

(c) Anja borrows \$NZ400 at 15% interest compounded annually at the start of each new year (there is no interest during the first year). She doesn't pay anything back during the first two years.

(i) How much does Anja owe altogether after the first amount of interest is added on?

(ii) How much does she owe altogether after the second amount of interest is added on?

(iii) During the third year Anja pays back \$NZ400. How much does she still owe after the third amount of interest is added on?

Question 2: 10 marks (Years 10 and below only)

Consider the number 854.

(a) Reverse the digits and find the difference between 854 and the number you found, taking the smaller number away from the larger one.

(b) Reverse the digits of the answer to (a) and add this number to the answer in (a). What result do you get?

(c) Repeat steps (a) and (b) with any three digit number *abc* of your your own choosing where $a > c + 1$. Do you get the same result as you found in part (b)?

(d) The number 854 can be written in expanded form as $8 \times 100 + 5 \times 10 + 4$. Write the general 3-digit number *abc* (where $a, b, c > 0$ and $a > c$) in expanded form.

(e) Now consider the 'reversed' number *cba* (where $a, b, c > 0$ and $a > c$). Write that in expanded form and subtract it away from the expanded form of *abc,* simplifying your answer as far as possible.

Question 3: 20 marks (All Years)

(a) Find the sum of the arithmetic series $T = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$. (An arithmetic series has a constant difference between consecutive terms.)

(b) If $a = 1$ and $z = 10$ are the first and last terms of the series T in (a) then $a + z = 11$. How many pairs of numbers altogether (including 1 and 10) in the series T add to 11?

(c) Write the answer to (a) as the product of two whole numbers.

(d) With brief working, hence or otherwise find the sum of the series $U = 1 + 2 + 3 + \cdots + 98 + 99 + 100$.

(e) True or false? (You do not have to show working.)

(i) The first and last terms of the series $V = 1 + 2 + 3 + \cdots + 48 + 49 + 50$ add to 51.

(ii) The sum of the series $V = 1 + 2 + 3 + \cdots + 48 + 49 + 50$ is 1275.

(iii) There are 501 pairs of numbers in the series $W = 1 + 2 + 3 + \cdots + 998 + 999 + 1000$ that add to 1001.

(iv) The sum of the series $W = 1 + 2 + 3 + \cdots + 998 + 999 + 1000$ is 500 500.

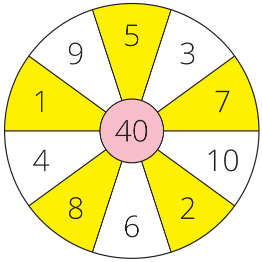
Now consider the general arithmetic series $S = a + \cdots + z$ which has exactly n terms.

(f) In terms of n how many pairs of terms of the series S add to $(a + z)$?

(g) Write down in terms of n, a, and z your best guess (based on this question) for the formula of the sum of n terms of the series S.

Question 4: 20 marks (All Years)

Mini-darts is a darts game for two junior players. The dartboard has 10 equal segments, numbered 1 to 10. If a dart lands in a segment the player earns the points for that segment. A central bulls-eye is worth 40, but missing the board earns 0. After tossing a coin to see who throws first each player in turn may throw up to three darts per visit to the board. They can land all three darts in the same scoring zone. The winner is the first to reach exactly 261 from as many visits to the dartboard as needed. If they go over 261 they lose all their points from that visit to the table and must sit down immediately.



(a) True or false? (You don't need to show working.)
  (i) The most you can score with three darts is 120.
  (ii) You can reach 261 in a minimum of three visits to the mini-dart board. (One visit is up to three throws of the darts at the board.)
  (iii) If you need 18 to win you can do it with two darts but one dart is not enough.

(b) Clint has three darts with which to score exactly 3 to win. What is the minimum number of throws he needs to win?

(c) In another game Clint has three darts to score exactly 21 to win. What is the minimum number of throws he needs to win?

(d) In a third game Clint has three throws to score exactly seven (7) to win.
  (i) If he misses the board completely with the first dart but reaches a score of seven with exactly two more throws in how many different ways can he throw the two darts without worrying about order?
  (ii) If Clint throws seven to win in exactly two throws (and so doesn't need the third dart) list as ordered pairs all the ways he could do this. For example (4, 3) is one way. This time the order of the throws is counted.
  (iii) If Clint needs all three darts (and none of them scores zero) list as triples all the ways he could do this. For example, (5, 1, 1) and (4, 1, 2) and (3, 2, 2) are ways Clint could score 7 points with three darts.

Question 5: 20 marks (All Years)

Many scientists believe that the Earth was formed about 4.5 billion years ago, where one billion is equal to 1000 million (and where one million = 1 000 000). In this question we will assume that this is correct. If the age of the earth were to be compressed into 24 hours then each hour would represent 187.5 million years in real time.

(a) Using this scale how many real time years would 12 compressed hours represent?

(b) Using this scale how many real time years would 20 compressed minutes represent?

(c) Explain mathematically in one line only how we worked out the number 187.5 million.

(d) If 4.5 billion = 4.5×10^9 then write 4.5 billion as an 'ordinary' number.

(e) If dinosaurs became extinct approximately 66 million years ago in real time, at what time on the 24 hour clock did dinosaurs become extinct on Earth? Give your answer to the nearest half-hour.

(f) If modern humans have been in existence for 200 000 years in real time at what time on the 24 hour clock did modern humans first exist on Earth? Give your answer to the nearest minute.

(g) If life began on Earth at 5 am in the morning on the 24 hour clock we are using in this question how many years ago in the real time life of the Earth did life begin?

(turn over)

Question 6: 20 marks (All Years)

A prime number has exactly two factors (1 and itself). The number 1 (which is not prime) only has one factor (namely itself), while composite numbers (all numbers that are not 1 and are not prime) have at least three factors. For example, 103 is prime, with factors 1 and 103. 30 has 1, 2, 3, 5, 6, 10, 15, and 30 as factors (so eight factors in total), while 25 has only three factors (namely 1, 5, and 25).

(a) Find the smallest three numbers with exactly 2, 3, and 4 factors respectively.

(b) Briefly explaining your answer (perhaps with a list), how many numbers less than 20 have

  (i) 2 factors?

  (ii) 3 factors?

  (iii) 4 factors?

  (iv) more than 4 factors?

(c) Briefly explain why the only numbers with exactly 3 factors have the form p^2, where p is a prime number.

(d) Can a number have exactly 5 factors? Briefly explain your answer, giving an example less than 100 if your answer is yes.

(e) A square-free number does not have any factors of the form p^2, where p is a prime number. Explain why every square-free number other than 1 has an even number of factors.

Question 7: 10 marks (Years 10 and 11 only)

Fahu has 12 identical (except for colour) cards: four red, four yellow, and four blue. He shuffles and draw three cards at random. Only one of three things can happen:

- Case 1: They're all the same colour.
- Case 2: They're all different colours.
- Case 3: Two are of one colour and one is of another.

Which of the three cases is the most likely? Show all necessary working. If your answer is correct but you do not show working you will receive no credit.

Question 8: 10 marks (Year 11 only)

Barbara the Builder has to build a backyard swimming pool for a rich client. There are four requirements:

- It must be rectangular.
- The depth must be 1.5 m.
- The surface area of the pool must be 275 m^2.
- The length of a diagonal (from corner to opposite corner) must be 25 m.

If x is the breadth of the pool and y is the length, requirements 3 and 4 lead to the two equations:

- equation A: $xy = 275$.
- equation B: $x^2 + y^2 = 625$.

(a) Make y the subject of equation A.

(b) Solve the equations simultaneously. Determine every real solution even if an answer contains negative results, giving your answers as ordered pairs.

(c) Give the dimensions of the feasible solution to Barbara's problem if x is less than y. Round your answers to one decimal place if necessary.

(END OF COMPETITION)