UNIVERSITY of OTAGO
Te Whare Wānanga o Otāgo
NEW ZEALAND

Department of Mathematics and Statistics

Junior Mathematics Competition 2017

Questions

TIME ALLOWED: ONE HOUR

Only Year 9 and below candidates may attempt QUESTION ONE

ALL candidates may attempt QUESTIONS TWO to FIVE

These questions are designed to test ability to analyse a problem and to express a solution clearly and accurately.

Please read the following instructions carefully before you begin:

(a) Do as much as you can. You are not expected to complete the entire paper. In the past, full answers to three questions have represented an excellent effort.

(b) You must explain your reasoning as clearly as possible, with a careful statement of the main points in the argument or the main steps in the calculation. Generally, even a correct answer without any explanation will not receive more than half credit. Likewise, clear and complete solutions to two problems will generally gain more credit than sketchy work on four.

(c) Credit will be given for partial solutions and evidence of a serious attempt to tackle a problem.

(d) Textbooks are NOT allowed. Calculators may be used and students who do not have one may be disadvantaged. Translation devices without computational capability are permitted provided any communications capability has been switched off. Otherwise normal examination conditions apply.

(e) Diagrams are a guide only and are not necessarily drawn to scale.

(f) We recommend black or blue pens. Dark pencil is acceptable if you have nothing else. Do NOT use red or green pens, or light pencil that we cannot read.

(g) We will penalise inappropriate rounding and incorrect or absent units.

(h) You do not lose marks for incorrect answers.

DO NOT TURN OVER UNTIL TOLD TO DO SO.

Question 1 (Year 9 and below only)

(a) Over a five day period at Kakanui during April the daily amount of rainfall was recorded.

Day:	1	2	3	4	5
Rainfall (mm):	3	0	4	26	5

(i) How much total rainfall was there over the five day period?
(ii) Show that the average rainfall per day over the five day period was 7.6 mm.

(b) Over the same month (30 days) the average rainfall per day at Kakanui was 5.6 mm.

(i) What was the total rainfall for April?
(ii) What was the average rainfall for the 25 days not included in the table in part (a)?

(c) On Day 2 the maximum temperature was 16°C (degrees Celsius). On Day 3 the maximum temperature was 10% less than that on Day 2. What was the maximum temperature on Day 3?

(d) Some countries measure temperature using the Fahrenheit scale, while New Zealand uses Celsius. The formula for converting Celsius (C) to Fahrenheit (F) is $F = \frac{9}{5}C + 32$.

(i) If the temperature in degrees Celsius is 20° what is the temperature in degrees Fahrenheit?
(ii) If the temperature in degrees Fahrenheit is 50° what is the temperature in degrees Celsius?
(iii) Find the temperature where $F = C$.

Question 2 (All Years)

A prime number has exactly two factors, 1 and itself. By this definition, 1 is not a prime number. The first fifteen prime numbers are

2; 3; 5; 7; 11; 13; 17; 19; 23; 29; 31; 37; 41; 43; 47

To check whether a number is prime or not, one method is:

(1) Use a calculator to find the square root of the number (for example $\sqrt{41}$ = 6.403 to three decimal places).

(2) Divide the number (x say) by all the prime numbers less than $\sqrt{x}$ in decreasing order.
For example for 41 you have to divide 41 by 5, divide 41 by 3, then divide 41 by 2. If it is divisible by any of these numbers then it is not a prime. However, you do not have to divide by 7 (or anything higher).

With this information answer the following questions:

(a) Germain primes are named after the great French mathematician Sophie Germain. A prime p is a Germain prime if the number $2p + 1$ is also a prime. For example, 41 is a Germain prime because 2 x 41 + 1 = 83, which is also prime.

Are the following statements true or false?

(i) 3 is a Germain prime.
(ii) 11 is a Germain prime.
(iii) 23 is a Germain prime.
(iv) 2017 is a Germain prime.

(b) 2017 is prime. Show that 2009 is not prime. Hint: Note that $\sqrt{2009}$ = 44.82 to 2 decimal places.

(c) There is only one prime between 2006 and 2016. State its value. You do not have to show working.

(d) We define whole numbers a and b to be *cousin primes* if a and b are primes and $a - b = 4$. For example 3 and 7 are cousin primes because 7 – 3 = 4.

(i) Find a pair of cousin primes where both are between 10 and 20.
(ii) Find a pair of cousin primes where both are between 100 and 120.
(iii) Is 2017 one of a pair of cousin primes? Show your reasoning.

Question 3 (All students)

In one episode of The Simpsons, the crowd at a baseball game was asked to guess the attendance. There were three options: 8128, 8191, and 8208. All three numbers are interesting from a mathematical point of view.

(a) The first, 8128, is a *perfect number*. This means that after you find all the factors of it and add them up (apart from 8128 itself), the result is 8128. The first 'perfect' number is 6, because when you add the factors of 6 (apart from 6 itself) you get 1 + 2 + 3 = 6.

(i) Write down in ascending order all the factors of 28 (apart from 28 itself), including 1.

(ii) Is 28 a perfect number? (You do not have to show working.)

(iii) Write down in ascending order all the factors of 8128 (apart from 8128 itself) including 1. (Hint: there are 13 factors, not counting 8128 itself. One of them is 64.)

(b) The second number, 8191, is a *Mersenne prime number*. A prime number has exactly two factors, 1 and itself, and a Mersenne prime has the form

$$2^p - 1$$

where p is a prime number. For example, if $p = 3$, the number $2^3 - 1 = 7$ is a Mersenne prime. Not all numbers of the form $2^p - 1$ are prime. For example, if $p = 23$ then $2^{23} - 1 = 8388607$, which is not prime since $8388607 = 47 \times 178481$.

(i) Solve for p the equation

$$2^p - 1 = 8191$$

You do not need to show working.

(ii) Show that $2^{11} - 1$ is not a prime number.

(c) The third number, 8208, is a *narcissistic number*. There are four digits and when you add their fourth powers it makes 8208. In other words

$$8^4 + 2^4 + 0^4 + 8^4 = 8208$$

Note that if there are two digits the power involved is 2. If there are three digits the power involved is three. So a three digit example is $370 = 3^3 + 7^3 + 0^3$

(i) Show whether 35 is narcissistic or not.

(ii) Show whether 153 is narcissistic or not.

(iii) Explain why there are no two digit narcissistic numbers starting with 1.

Question 4 (All Years)

In this question we are investigating the straight line $11x + 13y = 382$ (which we call equation *). A diagram of the line (not to scale) is shown in Figure 1.

(a) If we substitute $x = 0$ into the equation *, which point (A or B) do we find the co-ordinates of?

(b) Substitute $x = 0$ into the equation * to find the co-ordinates (in the form (a, b), where a and b are rational numbers) of the appropriate intercept.

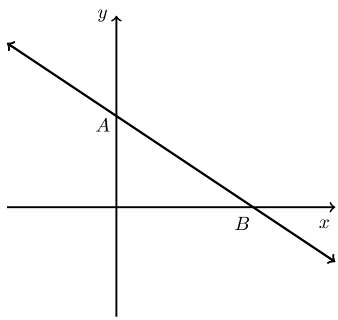

Figure 1 (not to scale)

(Question 4 continued on next page)

Question 4 (All Years) (continued)

A third variable t may be introduced to equation * to give the equivalent equation

$$11(2292 - 13t) + 13(-1910 + 11t) = 382$$

In this version of the equation $2292 - 13t = x$ and $-1910 + 11t = y$.

(c) If $t = 0$ write down the values of x and y.

(d) Show that if $t = 357$ then $y = 2017$ and find the corresponding value for x.

(e) Solve for t the two inequalities

$$2292 - 13t > 0$$

and

$$-1910 + 11t > 0$$

Give all integer values between the two solutions you have found.

(f) Use the solution to part (e) to find the co-ordinates of all points on the line * where x and y are both positive integers.

Question 5 (All Years)

Moana designs a stylised flower inside a circle for a company logo. Each petal consists of a semicircle with radius a joined to an isosceles triangle with two leg sides of length b, a base side of length c, and a vertex angle of 30°. (Note that $c = 2a$.) A circle of radius r is drawn around the flower such that the curved edge of eachpetal just touches the circle's circumference. See Figure 2.

(a) What is the centre angle of four adjacent petals combined?

(b) The logo in Figure 2 has 12 petals. If the vertex angle of each petal was 20° instead of 30°, how many petals would there be?

(c) Suppose each self-contained area of the logo is coloured in. Each self-contained area of the logo cannot have the same colour with an area adjacent to itself. How many different colours are needed? Briefly explain your answer.

For parts (d) and (e) assume $b = 3$ cm.

(d) (i) Show that the length of c is 1.553 cm (to 3 decimal places). (Use this value of c from now on if you cannot find it yourself.)

(ii) Hence find the area of an individual petal. Give your final answer to 3 decimal places but be as accurate as possible with your working.

(e) Find the area of the outer circle not covered by the flower. Give your final answer to 3 decimal places but be as accurate as possible with your working.

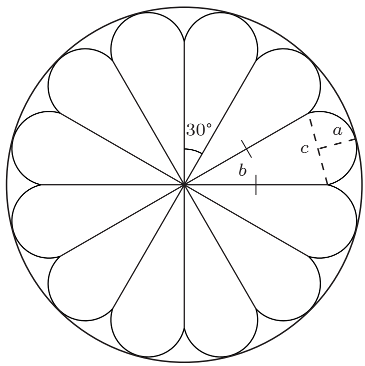

Figure 2 (not to scale)

(END OF COMPETITION)